Appendix

Proof of Observation 1

We consider any allocation rule $\mathcal{A}$. If the quantity produced exceeds demand (i.e., $q > d(p,\epsilon)$), then we have $CS^H(\epsilon) = CS^{\mathcal{A}}(\epsilon) = CS^L(\epsilon)$. We next consider the case when $q \le d(p,\epsilon)$. We have

$$\begin{aligned}
CS^H(\epsilon) - CS^{\mathcal{A}}(\epsilon) &= \int_0^{d(p,\epsilon)} \mathbb{1}_{\{w\le q\}}(d^{-1}(w,\epsilon)-p)\,dw - \int_0^{d(p,\epsilon)} \mathcal{A}(w)(d^{-1}(w,\epsilon)-p)\,dw \\
&= \int_0^{d(p,\epsilon)} (\mathbb{1}_{\{w\le q\}} - \mathcal{A}(w))(d^{-1}(w,\epsilon)-p)\,dw \\
&= \int_0^{q} (1-\mathcal{A}(w))(d^{-1}(w,\epsilon)-p)\,dw - \int_q^{d(p,\epsilon)} \mathcal{A}(w)(d^{-1}(w,\epsilon)-p)\,dw \\
&\ge \int_0^{q} (1-\mathcal{A}(w))(d^{-1}(q,\epsilon)-p)\,dw - \int_q^{d(p,\epsilon)} \mathcal{A}(w)(d^{-1}(q,\epsilon)-p)\,dw \\
&= (d^{-1}(q,\epsilon)-p)\big[q - \int_0^{d(p,\epsilon)} \mathcal{A}(w)\,dw\big] = (d^{-1}(q,\epsilon)-p)\big[q - d(p,\epsilon)\frac{q}{d(p,\epsilon)}\big] = 0.
\end{aligned}$$

We also have

$$\begin{aligned}
CS^{\mathcal{A}}(\epsilon) - CS^L(\epsilon) &= \int_0^{d(p,\epsilon)} \mathcal{A}(w)(d^{-1}(w,\epsilon)-p)\,dw - \int_0^{d(p,\epsilon)} \mathbb{1}_{\{w\ge d(p,\epsilon)-q\}}(d^{-1}(w,\epsilon)-p)\,dw \\
&= \int_0^{d(p,\epsilon)} (\mathcal{A}(w) - \mathbb{1}_{\{w\ge d(p,\epsilon)-q\}})(d^{-1}(w,\epsilon)-p)\,dw \\
&= \int_0^{d(p,\epsilon)-q} \mathcal{A}(w)(d^{-1}(w,\epsilon)-p)\,dw + \int_{d(p,\epsilon)-q}^{d(p,\epsilon)} (\mathcal{A}(w)-1)(d^{-1}(w,\epsilon)-p)\,dw \\
&\ge \int_0^{d(p,\epsilon)-q} \mathcal{A}(w)(d^{-1}(q,\epsilon)-p)\,dw + \int_{d(p,\epsilon)-q}^{d(p,\epsilon)} (\mathcal{A}(w)-1)(d^{-1}(q,\epsilon)-p)\,dw \\
&= (d^{-1}(q,\epsilon)-p)\big[\int_0^{d(p,\epsilon)} \mathcal{A}(w)\,dw - q\big] = (d^{-1}(q,\epsilon)-p)\big[d(p,\epsilon)\frac{q}{d(p,\epsilon)} - 1\big] = 0.
\end{aligned}$$

This concludes the proof.

Proof of Observation 2

We first show that $CS^{\mathcal{A}}(\boldsymbol{\epsilon}) \le CS^H(\boldsymbol{\epsilon})$. We divide the proof into two steps. After defining an auxiliary allocation $\mathcal{A}'$, we will show that $CS^{\mathcal{A}}(\boldsymbol{\epsilon}) \le CS^{\mathcal{A}'}(\boldsymbol{\epsilon})$ and then $CS^{\mathcal{A}'}(\boldsymbol{\epsilon}) \le CS^H(\boldsymbol{\epsilon})$. Consider a price vector $\mathbf{p}$, a quantity vector $\mathbf{q}$, and a noise realization $\boldsymbol{\epsilon} \in \Omega$. Let $\mathcal{C}^{\mathcal{A}}$ be a path associated with the allocation $\mathcal{A}$ and $\mathbf{r}^{\mathcal{A}}$ its parametric function. Let the allocation $\mathcal{A}'$ be such that $\mathcal{A}'_i(\mathbf{r}^{\mathcal{A}'}) = \mathbb{1}_{\{r_i^{\mathcal{A}'} \le \min\{d_i(\mathbf{p},\boldsymbol{\epsilon}), q_i\}\}}$ and $\mathbf{r}^{\mathcal{A}'} = \mathbf{r}^{\mathcal{A}}$.

We next show that $CS^{\mathcal{A}}(\boldsymbol{\epsilon}) \le CS^{\mathcal{A}'}(\boldsymbol{\epsilon})$. For simplicity, we assume that the path is non-decreasing in each component. For each $i \in \{1,\dots,n\}$, we partition the path $\mathcal{C}^{\mathcal{A}}$ into the two following sub-paths: $\mathcal{C}^{\mathcal{A}}_{1,i} = \{\mathbf{y} \in \mathbb{R}^n | \mathbf{y} = r^{\mathcal{A}}(w), r_i^{\mathcal{A}}(w) \le \min\{d_i(\mathbf{p},\boldsymbol{\epsilon}), q_i\}, w \in [a,b]\}$ and $\mathcal{C}^{\mathcal{A}}_{2,i} = \{\mathbf{y} \in \mathbb{R}^n | \mathbf{y} = r^{\mathcal{A}}(w), r_i^{\mathcal{A}}(w) > \min\{d_i(\mathbf{p},\boldsymbol{\epsilon}), q_i\}, w \in [a,b]\}$. Consider $\tilde{\mathbf{q}}^{\mathbf{i}} = \arg\min_{\mathbf{x}\in\mathcal{C}^{\mathcal{A}}_{1,i}} \{d_i^{-1}(\mathbf{x},\boldsymbol{\epsilon})\}$ (if the solution is not a singleton, one can select any of the solutions). We then have the following:

$$CS^{\mathcal{A}'}(\boldsymbol{\epsilon}) - CS^{\mathcal{A}}(\boldsymbol{\epsilon}) = \int_{\mathcal{C}^{\mathcal{A}}} \sum_{i=1}^n (d_i^{-1}(\mathbf{r}^{\mathcal{A}},\boldsymbol{\epsilon}) - p_i)\mathbb{1}_{\{r_i^{\mathcal{A}} \le \min\{d_i(\mathbf{p},\boldsymbol{\epsilon}),q_i\}\}} dr_i^{\mathcal{A}} - \int_{\mathcal{C}^{\mathcal{A}}} \sum_{i=1}^n (d_i^{-1}(\mathbf{r}^{\mathcal{A}},\boldsymbol{\epsilon}) - p_i)\mathcal{A}_i(\mathbf{r}^{\mathcal{A}}) dr_i^{\mathcal{A}}$$

$$= \sum_{i=1}^{n} \int_{\mathcal{C}_{1,i}^{\mathcal{A}}} (d_i^{-1}(\mathbf{r}^{\mathcal{A}}, \boldsymbol{\epsilon}) - p_i)(1 - \mathcal{A}_i(\mathbf{r}^{\mathcal{A}}))dr_i^{\mathcal{A}} - \sum_{i=1}^{n} \int_{\mathcal{C}_{2,i}^{\mathcal{A}}} (d_i^{-1}(\mathbf{r}^{\mathcal{A}}, \boldsymbol{\epsilon}) - p_i)\mathcal{A}_i(\mathbf{r}^{\mathcal{A}})dr_i^{\mathcal{A}}$$
$$\geq \sum_{i=1}^{n} \int_{\mathcal{C}_{1,i}^{\mathcal{A}}} (d_i^{-1}(\tilde{\mathbf{q}}^{\mathbf{i}}, \boldsymbol{\epsilon}) - p_i)(1 - \mathcal{A}_i(\mathbf{r}^{\mathcal{A}}))dr_i^{\mathcal{A}} - \sum_{i=1}^{n} \int_{\mathcal{C}_{2,i}^{\mathcal{A}}} (d_i^{-1}(\tilde{\mathbf{q}}^{\mathbf{i}}, \boldsymbol{\epsilon}) - p_i)\mathcal{A}_i(\mathbf{r}^{\mathcal{A}})dr_i^{\mathcal{A}}$$
$$= \sum_{i=1}^{n} \int_{\mathcal{C}_{1,i}^{\mathcal{A}}} (d_i^{-1}(\tilde{\mathbf{q}}^{\mathbf{i}}, \boldsymbol{\epsilon}) - p_i)dr_i^{\mathcal{A}} - \sum_{i=1}^{n} \int_{\mathcal{C}^{\mathcal{A}}} (d_i^{-1}(\tilde{\mathbf{q}}^{\mathbf{i}}, \boldsymbol{\epsilon}) - p_i)\mathcal{A}_i(\mathbf{r}^{\mathcal{A}})dr_i^{\mathcal{A}}$$
$$= \sum_{i=1}^{n} \int_{\mathcal{C}^{\mathcal{A}}} (d_i^{-1}(\tilde{\mathbf{q}}^{\mathbf{i}}, \boldsymbol{\epsilon}) - p_i)\mathbb{1}_{\{r_i^{\mathcal{A}} \leq \min\{d_i(\mathbf{p},\boldsymbol{\epsilon}), q_i\}\}}dr_i^{\mathcal{A}} - \sum_{i=1}^{n} \int_{\mathcal{C}^{\mathcal{A}}} (d_i^{-1}(\tilde{\mathbf{q}}^{\mathbf{i}}, \boldsymbol{\epsilon}) - p_i)\mathcal{A}_i(\mathbf{r}^{\mathcal{A}})dr_i^{\mathcal{A}}$$
$$= \sum_{i=1}^{n} \min\{q_i, d_i(\mathbf{p}, \boldsymbol{\epsilon})\}(d_i^{-1}(\tilde{\mathbf{q}}^{\mathbf{i}}, \boldsymbol{\epsilon}) - p_i) - \sum_{i=1}^{n} \min\{q_i, d_i(\mathbf{p}, \boldsymbol{\epsilon})\}(d_i^{-1}(\tilde{\mathbf{q}}^{\mathbf{i}}, \boldsymbol{\epsilon}) - p_i) = 0.$$

The above inequality follows from two facts. First, note that $d_i^{-1}(\mathbf{x}, \boldsymbol{\epsilon}) \geq d_i^{-1}(\tilde{\mathbf{q}}^{\mathbf{i}}, \boldsymbol{\epsilon})$ for $\mathbf{x} \in \mathcal{C}_{1,i}^{\mathcal{A}}$ by the definition of $\tilde{\mathbf{q}}^{\mathbf{i}}$. Second, $d_i^{-1}(\tilde{\mathbf{q}}^{\mathbf{i}}, \boldsymbol{\epsilon}) \geq d_i^{-1}(\mathbf{x}, \boldsymbol{\epsilon})$ for $\mathbf{x} \in \mathcal{C}_{2,i}^{\mathcal{A}}$ since $\mathbf{x} \geq \tilde{\mathbf{q}}^{\mathbf{i}}$ for all $\mathbf{x} \in \mathcal{C}^{\mathcal{A}_{2,i}}$ (as the parametric function is non-decreasing) and the inverse demand function is non-increasing in each of the quantities (recall that the Jacobian of the inverse demand is non-positive). Then, we conclude that $CS^{\mathcal{A}}(\boldsymbol{\epsilon}) \leq CS^{\mathcal{A}'}(\boldsymbol{\epsilon})$.

We next show that $CS^{\mathcal{A}'}(\boldsymbol{\epsilon}) \leq CS^{H}(\boldsymbol{\epsilon})$. For the H rule, consider the following path (which is consistent with this rule):

$$\mathbf{r}^{\mathbf{H}}(w) = \begin{cases} \min\{\mathbf{r}^{\mathcal{A}}(w), d(\mathbf{p}, \boldsymbol{\epsilon}), \mathbf{q}\} & \text{for } w \in [a, b], \\ \min\{d(\mathbf{p}, \boldsymbol{\epsilon}), \mathbf{q}\} + (w - b)\max\{\mathbf{0}, d(\mathbf{p}, \boldsymbol{\epsilon}) - \mathbf{q}\} & \text{for } w \in (b, b+1]. \end{cases}$$

Note that $\frac{d\mathbf{r}^{\mathbf{H}}(w)}{dw} = \frac{d\mathbf{r}^{\mathcal{A}}(w)}{dw}\mathbb{1}_{\{\mathbf{r}^{\mathcal{A}}(w) \leq \min\{d(\mathbf{p},\boldsymbol{\epsilon}),\mathbf{q}\}\}}$ for $w \in [a, b]$. Then, we have:

$$CS^{H}(\boldsymbol{\epsilon}) - CS^{A'}(\boldsymbol{\epsilon}) = \int_a^{b+1} \sum_i (d_i^{-1}(\mathbf{r}^{\mathbf{H}}(w), \boldsymbol{\epsilon}) - p_i)\mathbb{1}_{\{\mathbf{r}^{\mathbf{H}}(w) \leq \min\{d(\mathbf{p},\boldsymbol{\epsilon}),\mathbf{q}\}\}}\frac{dr_i^H(w)}{dw}dw$$
$$- \int_a^b \sum_i (d_i^{-1}(\mathbf{r}^{\mathcal{A}}(w), \boldsymbol{\epsilon}) - p_i)\mathbb{1}_{\{r_i^{\mathcal{A}}(w) \leq \min\{d_i(\mathbf{p},\boldsymbol{\epsilon}),q_i\}\}}\frac{dr_i^A(w)}{dw}dw$$
$$= \int_a^b \sum_i (d_i^{-1}(\mathbf{r}^{\mathbf{H}}(w), \boldsymbol{\epsilon}) - p_i)\mathbb{1}_{\{\mathbf{r}^{\mathbf{H}}(w) \leq \min\{d(\mathbf{p},\boldsymbol{\epsilon}),\mathbf{q}\}\}}\frac{dr_i^H(w)}{dw}dw$$
$$+ \int_b^{b+1} \sum_i (d_i^{-1}(\mathbf{r}^{\mathbf{H}}(w), \boldsymbol{\epsilon}) - p_i)\mathbb{1}_{\{\mathbf{r}^{\mathbf{H}}(w) \leq \min\{d(\mathbf{p},\boldsymbol{\epsilon}),\mathbf{q}\}\}}\frac{dr_i^H(w)}{dw}dw$$
$$- \int_a^b \sum_i (d_i^{-1}(\mathbf{r}^{\mathcal{A}}(w), \boldsymbol{\epsilon}) - p_i)\mathbb{1}_{\{r_i^{\mathcal{A}}(w) \leq \min\{d_i(\mathbf{p},\boldsymbol{\epsilon}),q_i\}\}}\frac{dr_i^A(w)}{dw}dw$$
$$= \int_a^b \sum_i (d_i^{-1}(\mathbf{r}^{\mathbf{H}}(w), \boldsymbol{\epsilon}) - p_i)\frac{dr_i^H(w)}{dw}dw$$
$$- \int_a^b \sum_i (d_i^{-1}(\mathbf{r}^{\mathcal{A}}(w), \boldsymbol{\epsilon}) - p_i)\mathbb{1}_{\{r_i^{\mathcal{A}}(w) \leq \min\{d_i(\mathbf{p},\boldsymbol{\epsilon}),q_i\}\}}\frac{dr_i^A(w)}{dw}dw$$
$$= \int_a^b \sum_i (d_i^{-1}(\min\{\mathbf{r}^{\mathbf{A}}(w), d(\mathbf{p}, \boldsymbol{\epsilon}), \mathbf{q}\}) - p_i)\mathbb{1}_{\{r_i^{A}(w) \leq \min\{d_i(\mathbf{p},\boldsymbol{\epsilon}),q_i\}\}}\frac{dr_i^A(w)}{dw}dw$$
$$- \int_a^b \sum_i (d_i^{-1}(\mathbf{r}^{\mathcal{A}}(w), \boldsymbol{\epsilon}) - p_i)\mathbb{1}_{\{r_i^{\mathcal{A}}(w) \leq \min\{d_i(\mathbf{p},\boldsymbol{\epsilon}),q_i\}\}}\frac{dr_i^A(w)}{dw}dw \geq 0.$$

This concludes the proof. Note that the proof for $CS^L(\boldsymbol{\epsilon}) \leq CS^{\mathcal{A}}(\boldsymbol{\epsilon})$ is analogous and, hence, we omit it for conciseness.

Limit Argument for the R Rule

We motivate the interpretation of the R rule as the limit case of the H rule. We start by considering the case of the single product, and then proceed with the multiple-product setting. For illustration purposes, we consider a demand realization such that $d = 2q$. Under the H rule, the items are allocated to customers in the first half (i.e., with highest valuations), see top-left panel of figure 6. Next, we divide customer demand into two halves and apply the H rule to each half. In this case, only the first and third quarters of customers (according to their valuations) will receive the product, see the middle-left panel in Figure 6. Then, we can proceed by doing the same process of applying the H rule to each quarter of customers, see the bottom-left panel of Figure 6. We continue the process of dividing demand into a larger number of groups composed of non-atomic customers and apply the H rule within each group. In the limit, this process converges to the R rule. The left panels of Figure 6 represent the Consumer Surplus when applying the H rule for one, two, and four sub-segments of the consumers in the top, middle, and bottom panels respectively. When increasing the number of subdivisions, N, the Consumer Surplus ends up equating the expression given in Equation (4). A formal proof of the latter statement is provided later in this appendix.

The same intuition holds for the setting with multiple products. To simplify the illustration, we consider $n = 2$ products such that $d_1 = 2q_1$ and $d_2 = q_2$. We start allocating items with the H rule so that the representative consumer is given q_1, q_2 leaving unserved the demand from (q_1, q_2) to $d = (2q_1, q_2)$ (see top-right panel of Figure 6). We next split the demand into two halves, so that we apply the H rule again but within each half, namely, between $\mathbf{0}$ and $\mathbf{d}/2 = (q_1, q_2/2)$ we give $q_1/2$ and $q_2/2$ leaving the second quarter of demand for item 1 unmet, and similarly for the second half. Mathematically, this would be like integrating over a path that goes from $\mathbf{0}$ to $\mathbf{q}/2$ weighting the inverse demand by 1, and then over a path from $\mathbf{q}/2$ to $\mathbf{d}/2$ with a weight of 0 (see the middle-right panel of Figure 6). Similarly, for the second half of the demand, we integrate over a path from $\mathbf{d}/2$ to $\mathbf{d}/2 + \mathbf{q}/2$ with a weight of 1 and from $\mathbf{d}/2 + \mathbf{q}/2$ to $\mathbf{d}$ with a weight of 0. We then continue iteratively to divide the demand by N groups and consider the case where N goes to infinity. We thus end up with the R rule.

5.1. Proof of the Limit Argument for the R Rule

Let $\mathbf{r^k}$ be the sub-path that goes through a straight line (without loss of generality) from $\frac{(K-1)}{N} d_i(\mathbf{p}, \boldsymbol{\epsilon})$ to $\frac{(K-1)}{N} d_i(\mathbf{p}, \boldsymbol{\epsilon}) + \frac{\min\{(\mathbf{p},\boldsymbol{\epsilon}),\mathbf{q}\}}{N}$ and then to $\frac{k}{N} d_i(\mathbf{p}, \boldsymbol{\epsilon})$ for $k \in \{1, \dots, N\}$. Let $\mathbf{r}$ be the path defined for the R rule. To simplify notation, we use d instead of $d(\mathbf{p}, \boldsymbol{\epsilon})$. One can write

$$\lim_{N\to\infty} \quad \sum_{k=1}^{N} \int_{\mathcal{C}_k} \sum_{i=1}^{n} (d_i^{-1}(\mathbf{r^k}, \boldsymbol{\epsilon}) - p_i) \mathcal{A}_i^k(\mathbf{r^k}) dr_i^k$$

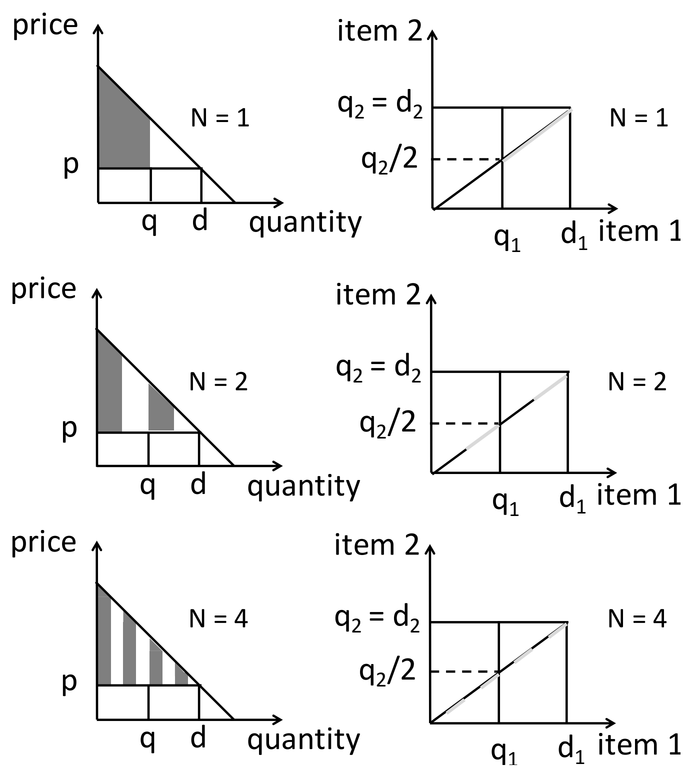

Figure 6 **Left: Consumer surplus of the H rule applied to one (top panel), two (middle panel), and four (bottom panel) segments of consumers. Right: The Consumer Surplus given as the integral of the marginal utility (or inverse demand function) along the dashed segment marked in black dividing the space of consumers into one, two, and four parts in the top, middle, and bottom panel respectively.**

$$
\begin{aligned}
&= \lim_{N\to\infty}\sum_{k=1}^{N}\int_{\mathcal{C}_k}\sum_{i=1}^{n}(d_i^{-1}(\mathbf{r^k},\boldsymbol{\epsilon})-p_i)\mathbb{1}_{\left\{\mathbf{r^k}\in\prod_{j=1}^{n}\left[\frac{(k-1)}{N}d_i,\frac{k-1}{N}d_i+\frac{\min\{d_i,q_i\}}{N}\right]\right\}}dr_i^k\\
&= \lim_{N\to\infty}\sum_{k=1}^{N}\int_{\mathcal{C}_k}\sum_{i=1}^{n}(d_i^{-1}(\mathbf{r^k},\boldsymbol{\epsilon})-p_i)dr_i^k\\
&= \lim_{N\to\infty}\sum_{k=1}^{N}\int_0^1\sum_{i=1}^{n}\left(d_i^{-1}\left(\frac{(k-1)}{N}d+\frac{w}{N}\min\{\mathbf{d},\mathbf{q}\},\boldsymbol{\epsilon}\right)-p_i\right)\min\{d_i,q_i\}dw\frac{1}{N}\\
&= \int_0^1\int_0^1\sum_{i=1}^{n}(d_i^{-1}(x\mathbf{d},\boldsymbol{\epsilon})-p_i)\min\{d_i,q_i\}dw\,dx = \int_0^1\sum_{i=1}^{n}(d_i^{-1}(x\mathbf{d},\boldsymbol{\epsilon})-p_i)\min\{d_i,q_i\}dx\\
&= \int_{\mathcal{C}^R}\sum_{i=1}^{n}(d_i^{-1}(\mathbf{r},\boldsymbol{\epsilon})-p_i)\min\{1,q_i/d_i\}dr_i = CS^R(\boldsymbol{\epsilon}).
\end{aligned}
$$

Proof of Proposition 2

We next show the claim for each of the three allocation rules.

- **H rule**

$$\begin{aligned}\mathbb{E}[CS^H(\epsilon)] &= \mathbb{E}\left[\int_0^{d(p^0)+\epsilon} \mathbb{1}_{\{w\le q^{sto}\}}(d^{-1}(w-\epsilon)-p^0)\,dw\right]\\ &\ge \mathbb{E}\left[\int_0^{d(p^0)+\epsilon} \mathbb{1}_{\{w\le d(p^0)\}}(d^{-1}(w-\epsilon)-p^0)\,dw\right]\\ &= \mathbb{E}\left[\mathbb{1}_{\{\epsilon\le 0\}}\int_0^{d(p^0)+\epsilon}(d^{-1}(w-\epsilon)-p^0)\,dw + \mathbb{1}_{\{\epsilon>0\}}\int_0^{d(p^0)}(d^{-1}(w-\epsilon)-p^0)\,dw\right].\end{aligned}$$

We denote by $K(\epsilon)$ the argument of the expectation in the last equation. We next show that $K(\epsilon)$ is convex. The first and second derivatives of the first term of $K(\epsilon)$ are given by:

$$\begin{aligned}\frac{d}{d\epsilon}\int_0^{d(p^0)+\epsilon}(d^{-1}(w-\epsilon)-p^0)\,dw &= d^{-1}(d(p^0))-p^0-\int_0^{d(p^0)+\epsilon}\frac{dd^{-1}(w-\epsilon)}{dx}\,dw\\ &= -\int_{-\epsilon}^{d(p^0)}\frac{dd^{-1}(x)}{dx}\,dx \qquad (x=w-\epsilon)\\ &= d^{-1}(-\epsilon)-p^0,\end{aligned}$$

$$\frac{d^2}{d\epsilon^2}\int_0^{d(p^0)+\epsilon}(d^{-1}(w-\epsilon)-p^0)\,dw = -\frac{dd^{-1}(-\epsilon)}{dx}\ge 0.$$

Regarding the second term of $K(\epsilon)$, we have

$$\begin{aligned}\frac{d}{d\epsilon}\int_0^{d(p^0)}(d^{-1}(w-\epsilon)-p^0)\,dw &= -\int_0^{d(p^0)+\epsilon}\frac{dd^{-1}(w-\epsilon)}{dx}\,dw\\ &= -\int_{-\epsilon}^{d(p^0)-\epsilon}\frac{dd^{-1}(x)}{dx}\,dx \qquad (x=w-\epsilon)\\ &= -d^{-1}(d(p^0)-\epsilon)+d^{-1}(-\epsilon),\end{aligned}$$

$$\frac{d^2}{d\epsilon^2}\int_0^{d(p^0)}(d^{-1}(w-\epsilon)-p^0)\,dw = \frac{d\,d^{-1}(d(p^0)-\epsilon)}{dx}-\frac{d\,d^{-1}(-\epsilon)}{dx}\ge 0.$$

Note that the last inequality follows from the convexity of $d(p)$. Thus, $K(\epsilon)$ is convex for both $\epsilon\le 0$ and $\epsilon>0$. In addition, the derivatives of both terms coincide when $\epsilon=0$ and, hence, $K(\epsilon)$ is convex. One can now use Jensen's inequality to conclude the proof:

$$\mathbb{E}[CS^H(\epsilon)] = \mathbb{E}\left[\int_0^{d(p^0)+\epsilon}\mathbb{1}_{\{w\le q^{sto}\}}(d^{-1}(w-\epsilon)-p^0)\,dw\right] \ge \int_0^{d(p^0)}(d^{-1}(w)-p^0)\,dw = CS_{det}.$$

- **L rule**

$$\begin{aligned}CS^L(\epsilon) &= \int_0^{d(p^0,\epsilon)}(d^{-1}(w,\epsilon)-p^0)\mathbb{1}_{\{w\ge d(p^0,\epsilon)-q^{sto}\}}\,dw = \int_0^{d(p^0)+\epsilon}(d^{-1}(w-\epsilon)-p^0)\mathbb{1}_{\{w\ge d(p^0)+\epsilon-q^{sto}\}}\,dw\\ &\le \int_0^{d(p^0)+\epsilon}(d^{-1}(w-\epsilon)-p^0)\mathbb{1}_{\{w\ge\epsilon\}}\,dw = \int_{\max\{0,\epsilon\}}^{d(p^0)+\epsilon}(d^{-1}(w-\epsilon)-p^0)\,dw\\ &= \int_{\max\{-\epsilon,0\}}^{d(p^0)}(d^{-1}(v)-p^0)\,dv \le \int_0^{d(p^0)}(d^{-1}(v)-p^0)\,dv = CS_{det}.\end{aligned}$$

Therefore, for any ϵ we have $CS^L(\epsilon) \leq CS_{det}$ (as long as $q^{sto} \leq d(p^0)$). Taking the expectation over ϵ, we obtain $\mathbb{E}[CS^L(\epsilon)] \leq CS_{det}$.

- **Random rule**

We first state and prove the following lemma.

LEMMA 3. *Consider $a > 0$ and $f(\cdot)$ be a non-negative concave function. Then, the function $h(x) = \frac{1}{a+x}\int_0^{a+x} f(w-x)dw$ is also concave (for $x > -a$).*

Proof. We compute the first and second derivatives of $h(x)$:

$$\begin{aligned}
\frac{dh(x)}{dx} &= \frac{f(a)}{a+x} - \frac{1}{(a+x)^2}\int_0^{a+x} f(w-x)dw - \frac{1}{a+x}\int_0^{a+x} f'(w-x)dw \\
&= \frac{f(a)}{a+x} - \frac{1}{(a+x)^2}\int_0^{a+x} f(w-x)dw - \frac{f(a)}{a+x} + \frac{1}{a+x}f(-x) \\
&= -\frac{1}{(a+x)^2}\int_0^{a+x} f(w-x)dw + \frac{f(-x)}{a+x},
\end{aligned}$$

$$\begin{aligned}
\frac{d^2h(x)}{dx^2} &= -\frac{f(a)}{(a+x)^2} + \frac{2}{(a+x)^3}\int_0^{a+x} f(w-x)dw \\
&\quad + \frac{1}{(a+x)^2}\int_0^{a+x} f'(w-x)dw - \frac{f'(-x)}{a+x} - \frac{f(-x)}{(a+x)^2} \\
&= -\frac{f(a)}{(a+x)^2} + \frac{2}{(a+x)^3}\int_0^{a+x} f(w-x)dw + \frac{f(a)}{(a+x)^2} - \frac{f(-x)}{(a+x)^2} \\
&\quad - \frac{f'(-x)}{a+x} - \frac{f(-x)}{(a+x)^2} \\
&= \frac{2}{(a+x)^3}\int_0^{a+x} f(w-x)dw - \frac{2f(-x)}{(a+x)^2} - \frac{f'(-x)}{a+x} \\
&\leq \frac{2}{(a+x)^3}\int_0^{a+x} f(-x) + f'(-x)w\, dw - \frac{2f(-x)}{(a+x)^2} - \frac{f'(-x)}{a+x} = 0.
\end{aligned}$$

The above inequality follows from $f(y) \leq f(x) + f'(x)(y-x)$ (given that f is concave). Note that alternatively, one can show that the function $h(x) = \frac{1}{x}\int_0^x f(z)dz$ is concave for $x > 0$. □

Using the R allocation rule, we have the following:

$$\begin{aligned}
\mathbb{E}[CS^R(\epsilon)] &= \mathbb{E}\left[\int_0^{d(p^0,\epsilon)} \min\left\{1, \frac{q^{sto}}{d(p^0,\epsilon)}\right\}\left(d^{-1}(w,\epsilon) - p^0\right) dw\right] \\
&= \mathbb{E}\left[\int_0^{d(p^0)+\epsilon} \min\left\{1, \frac{q^{sto}}{d(p^0)+\epsilon}\right\}\left(d^{-1}(w-\epsilon) - p^0\right) dw\right] \\
&\leq \mathbb{E}\left[\int_0^{d(p^0)+\epsilon} \min\left\{1, \frac{d(p^0)}{d(p^0)+\epsilon}\right\}\left(d^{-1}(w-\epsilon) - p^0\right) dw\right] \\
&\leq \mathbb{E}\left[\int_0^{d(p^0)+\epsilon} \frac{d(p^0)}{d(p^0)+\epsilon}(d^{-1}(w-\epsilon) - p^0)\, dw\right] \\
&\leq \int_0^{d(P)} \frac{d(p^0)}{d(p^0)}(d^{-1}(w) - P)\, dw = CS_{det}.
\end{aligned}$$

The first inequality follows from $q^{sto} \leq d(p^0)$ and the second inequality is obtained by dropping the minimum. The third inequality is a consequence of Lemma 3 by noting that $d^{-1}(x) - p^0$ is a non-negative concave function, i.e., we apply Lemma 3 for $f(y) = d^{-1}(y) - p^0$, $a = d(p^0)$, and $x = \epsilon$. Finally, the last inequality follows from Jensen's inequality.

Proof of Proposition 3

We consider the H allocation rule (which is the best scenario in terms of Consumer Surplus). Recall that (p^s, q^s) are the optimal prices and quantities from problem (6) and p^d is the optimal solution from problem (5). We then have

$$\begin{aligned}
\mathbb{E}[CS^H(\epsilon)] &= \mathbb{E}\left[\int_0^{d(p^s,\epsilon)} (d^{-1}(w,\epsilon) - p^s)\mathbb{1}_{\{w \leq q^s\}}\, dw\right] \\
&= \mathbb{E}\left[\int_0^{\epsilon d(p^s)} \left(d^{-1}\left(\frac{w}{\epsilon}\right) - p^s\right)\mathbb{1}_{\{w \leq q^s\}}\, dw\right] \\
&= \mathbb{E}\left[\int_0^{d(p^s)} (d^{-1}(v) - p^s)\mathbb{1}_{\{v\epsilon \leq q^s\}}\epsilon\, dv\right] \\
&\leq \mathbb{E}\left[\int_0^{d(p^s)} (d^{-1}(v) - p^s)\epsilon\, dv\right] = \int_0^{d(p^s)} (d^{-1}(v) - p^s)\, dv \\
&\leq \int_0^{d(p^d)} (d^{-1}(v) - p^d)\, dv = CS_{det},
\end{aligned}$$

where the second to last equality follows from $\mathbb{E}[\epsilon] = 1$ and the last inequality follows from the fact that $p^d \leq p^s$ for a multiplicative noise (see Lemma 1), plus the fact that the Consumer Surplus in the deterministic case is decreasing with the price. The last inequality follows from

$$\frac{d}{dp}\left[\int_0^{d(p)} (d^{-1}(w) - p)\, dw\right] = \int_0^{d(p)} -1\, dv = -d(p) \leq 0.$$

Proof of Proposition 4

We have

$$\begin{aligned}
\mathbb{E}[CS^H(\epsilon)] &= \mathbb{E}\left[\int_0^{d(p^s,\epsilon)} (d^{-1}(w,\epsilon) - p^s)\mathbb{1}_{\{w \leq q^s\}}\, dw\right] \\
&\geq \mathbb{E}\left[\int_0^{d(p^d,\epsilon)} (d^{-1}(w,\epsilon) - p^d)\mathbb{1}_{\{w \leq q^s\}}\, dw\right] \\
&\geq \mathbb{E}\left[\int_0^{d(p^d,\epsilon)} (d^{-1}(w,\epsilon) - p^d)\mathbb{1}_{\{w \leq d(p^d)\}}\, dw\right] \\
&= \mathbb{E}\left[\int_0^{d(p^d)+\epsilon} (d^{-1}(w-\epsilon) - p^d)\mathbb{1}_{\{w \leq d(p^d)\}}\, dw\right] \\
&\geq \int_0^{d(p^d)} (d^{-1}(w) - p^d)\, dw = CS_{det}.
\end{aligned}$$

The first inequality follows because the expression inside the expectation is decreasing with p^s:

$$\begin{aligned}&\frac{d}{dp^s}\int_0^{d(p^s,\epsilon)}(d^{-1}(w,\epsilon)-p^s)\mathbb{1}_{\{w\le q^s\}}\,dw\\ &=\frac{\partial d(p^s,\epsilon)}{\partial p^s}(d^{-1}(d(p^s,\epsilon),\epsilon)-p^s)\mathbb{1}_{\{w\le q^s\}}+\int_0^{d(p^s,\epsilon)}-\mathbb{1}_{\{w\le q^s\}}\,dw\\ &=-\min\{d(p^s,\epsilon),q^s\}\le 0.\end{aligned}$$

The second inequality is due to $q^s\ge d(p^s)$ and $d(p^s)\ge d(p^d)$ since $p^s\le p^d$ by Lemma 1. Finally, the third inequality follows from the analogous result under exogenous prices.

Proof of Proposition 5

We next show that $\mathbb{E}[CS^H(\epsilon)]\le CS_{det}$. Consider any path consistent with the H rule and $\mathbf{r}^\epsilon$ its parametric function. We then have

$$\begin{aligned}\mathbb{E}[CS^H(\epsilon)]&=\mathbb{E}\left[\int_{\mathcal{C}^\epsilon}\sum_i(d_i^{-1}(\mathbf{r}^\epsilon,\boldsymbol{\epsilon})-p_i^0)\mathbb{1}_{\{\mathbf{r}^\epsilon\le\min\{d(\mathbf{p^0},\epsilon),\mathbf{q^{sto}}\}\}}dr_i^\epsilon\right]\\ &=\mathbb{E}\left[\int_{\mathcal{C}^\epsilon}\sum_i(d_i^{-1}\left(\frac{\mathbf{r}^\epsilon}{\epsilon}\right)-p_i^0)\mathbb{1}_{\{\mathbf{r}^\epsilon\le\min\{d(\mathbf{p^0},\boldsymbol{\epsilon}),\mathbf{q^{sto}}\}\}}dr_i^\epsilon\right]\\ &=\mathbb{E}\left[\int_0^1\sum_i\left(d_i^{-1}(w\min\{\epsilon d(\mathbf{p^0}),\mathbf{q^{sto}}\}/\epsilon)-p_i^0\right)\min\{\epsilon d_i(\mathbf{p^0}),q_i^{sto}\}dw\right]\\ &=\mathbb{E}\left[\int_0^1\sum_i\left(d_i^{-1}\left(wd(\mathbf{p^0})\min\left\{1,\frac{1}{\epsilon}\right\}\right)-p_i^0\right)d_i(\mathbf{p^0})\epsilon\min\{1,1/\epsilon\}\,dw\right]\\ &=\mathbb{E}\left[\int_0^{\min\{1,1/\epsilon\}}\sum_i(d_i^{-1}(vd(\mathbf{p^0}))-p_i^0)d_i(\mathbf{p^0})\epsilon dv\right]\\ &\le\int_0^1\sum_i(d_i^{-1}(vd(\mathbf{p^0}))-p_i^0)d_i(\mathbf{p^0})dv=CS_{det}.\end{aligned}$$

Proof of Proposition 6

- **R rule.** Denote by $\mathcal{C}^\epsilon$ the path under the R rule and by r^ϵ its parametric function. We have

$$\begin{aligned}\mathbb{E}[CS^R(\boldsymbol{\epsilon})]&=\mathbb{E}\left[\int_{\mathcal{C}^\epsilon}\sum_{i=1}^n(d_i^{-1}(r^\epsilon,\boldsymbol{\epsilon})-p_i^0)\min\left\{1,\frac{q_i^s}{d_i(\mathbf{p^0},\boldsymbol{\epsilon})}\right\}dr_i^\epsilon\right]\\ &=\mathbb{E}\left[\int_{\mathcal{C}^\epsilon}\sum_{i=1}^n(d_i^{-1}(r^\epsilon-\boldsymbol{\epsilon})-p_i^0)\min\left\{1,\frac{q_i^s}{d_i(\mathbf{p^0})+\epsilon_i}\right\}dr_i^\epsilon\right]\\ &\le\mathbb{E}\left[\int_{\mathcal{C}^\epsilon}\sum_{i=1}^n(d_i^{-1}(r^\epsilon-\boldsymbol{\epsilon})-p_i^0)\min\left\{1,\frac{d_i(\mathbf{p^0})}{d_i(\mathbf{p^0})+\epsilon_i}\right\}dr_i^\epsilon\right]\\ &=\mathbb{E}\left[\int_{\mathcal{C}}\sum_{i=1}^n(d_i^{-1}(r-(\mathbf{I}-\mathbf{D^{-1}}_{d(\mathbf{p^0})}\mathbf{D}_r)\boldsymbol{\epsilon})-p_i^0)\right.\\ &\qquad\left.\min\left\{1,\frac{d_i(\mathbf{p^0})}{d_i(\mathbf{p^0})+\epsilon_i}\right\}\frac{d_i(\mathbf{p^0})+\epsilon_i}{d_i(\mathbf{p^0})}dr_i\right]\end{aligned}$$

$$= \mathbb{E}\left[\int_{\mathcal{C}} \sum_{i=1}^{n} (d_i^{-1}(r - (\mathbf{I} - \mathbf{D^{-1}}_{d(\mathbf{p^0})}\mathbf{D}_r)\boldsymbol{\epsilon}) - p_i^0) \min\left\{\frac{d_i(\mathbf{p^0}) + \epsilon_i}{d_i(\mathbf{p^0})}, 1\right\} dr_i\right]$$
$$\leq \mathbb{E}\left[\int_{\mathcal{C}} \sum_{i=1}^{n} (d_i^{-1}(r - (\mathbf{I} - \mathbf{D^{-1}}_{d(\mathbf{p^0})}\mathbf{D}_r)\boldsymbol{\epsilon}) - p_i^0) dr_i\right]$$
$$= \int_{\mathcal{C}} \sum_{i=1}^{n} \mathbb{E}\left[(d_i^{-1}(r - (\mathbf{I} - \mathbf{D^{-1}}_{d(\mathbf{p^0})}\mathbf{D}_r)\boldsymbol{\epsilon}) - p_i^0)\right] dr_i$$
$$\leq \int_{\mathcal{C}} \sum_{i=1}^{n} (d_i^{-1}(r) - p_i^0) dr_i = CS_{det}.$$

The second equality follows from the additive noise assumption and the first inequality from $\mathbf{q^s} \leq d(\mathbf{p^0})$. The third equality is obtained by the change of variable $r_i = r_i^{\boldsymbol{\epsilon}} \cdot d_i(p^0)/(d_i(p^0) + \epsilon_i)$. The last inequality follows from Jensen's inequality, given that we next show the concavity of $d_i^{-1}(r - (\mathbf{I} - \mathbf{D^{-1}}_{d(\mathbf{p^0})}\mathbf{D}_r)\boldsymbol{\epsilon})$ in $\boldsymbol{\epsilon}$, for all $i \in \{1, \ldots, n\}$. Indeed, we have

$$\nabla^2_{\boldsymbol{\epsilon}} d_i^{-1}\big(r - (\mathbf{I} - \mathbf{D^{-1}}_{d(\mathbf{p^0})}\mathbf{D}_r)\boldsymbol{\epsilon}\big) =$$
$$(\mathbf{I} - \mathbf{D^{-1}}_{d(\mathbf{p^0})}\mathbf{D}_r)\nabla^2 d_i^{-1}(r - (\mathbf{I} - \mathbf{D^{-1}}_{d(\mathbf{p^0})}\mathbf{D}_r)\boldsymbol{\epsilon})(\mathbf{I} - \mathbf{D^{-1}}_{d(\mathbf{p^0})}\mathbf{D}_r).$$

Note that the above expression is negative semi-definite as $\nabla^2 d_i^{-1}$ is assumed to be negative semi-definite. This concludes the proof.

- **L rule.** Let $\mathcal{C}^\epsilon$ be a consistent path with the L rule and r^ϵ its parametric function, then:

$$\begin{aligned}
\mathbb{E}[CS^L(\boldsymbol{\epsilon})] &= \int_{\mathcal{C}^\epsilon} \sum_i (d_i^{-1}(r^\epsilon, \boldsymbol{\epsilon}) - p_i^0)\mathbb{1}_{\{r^\epsilon \geq \max\{d(\mathbf{p^0},\boldsymbol{\epsilon}) - \mathbf{q^s}, \mathbf{0}\}\}} dr^\epsilon \\
&= \int_{\mathcal{C}^\epsilon} \sum_i (d_i^{-1}(r^\epsilon - \boldsymbol{\epsilon}) - p_i^0)\mathbb{1}_{\{r^\epsilon \geq \max\{d(\mathbf{p^0},\boldsymbol{\epsilon}) - \mathbf{q^s}, \mathbf{0}\}\}} dr^\epsilon \\
&\leq \int_{\mathcal{C}^\epsilon} \sum_i (d_i^{-1}(r^\epsilon - \boldsymbol{\epsilon}) - p_i^0)\mathbb{1}_{\{r^\epsilon \geq \max\{d(\mathbf{p^0},\boldsymbol{\epsilon}) - d(\mathbf{p^0}), \mathbf{0}\}\}} dr^\epsilon \\
&= \int_0^1 \sum_i (d_i^{-1}(\max\{\boldsymbol{\epsilon}, \mathbf{0}\} + w(d(\mathbf{p^0}) + \boldsymbol{\epsilon} - \max\{\boldsymbol{\epsilon}, \mathbf{0}\}) - \boldsymbol{\epsilon}) - p_i)(d_i(\mathbf{p^0}) + \epsilon_i - \max\{\epsilon_i, 0\}) dw \\
&= \int_0^1 \sum_i (d_i^{-1}(d(\mathbf{p^0})w - \min\{\mathbf{0}, \boldsymbol{\epsilon}\}(1-w)) - p_i^0)(d_i(\mathbf{p^0}) + \min\{0, \epsilon_i\}) dw \\
&\leq \int_0^1 \sum_i (d_i^{-1}(d(\mathbf{p^0})w) - p_i) d(\mathbf{p^0}) dw = CS_{det}.
\end{aligned}$$

In the fourth equality, we use the path that goes from $\mathbf{0}$ to $\max\{d(\mathbf{p^0}, \boldsymbol{\epsilon}) - \mathbf{q^s}, \mathbf{0}\}$, and then goes straight from $\max\{d(\mathbf{p^0}, \boldsymbol{\epsilon}) - \mathbf{q^s}, \mathbf{0}\}$ to $d(\mathbf{p^0}, \boldsymbol{\epsilon})$, namely, we set $r^\epsilon(w) = \max\{d(\mathbf{p^0}, \boldsymbol{\epsilon}) - \mathbf{q^s}, \mathbf{0}\} + (d(\mathbf{p^0}, \boldsymbol{\epsilon}) - \max\{d(\mathbf{p^0}, \boldsymbol{\epsilon}) - \mathbf{q^s}, \mathbf{0}\})w$ for $w \in [0, 1]$ for the second part of the path. The second equality holds as $d^{-1}(\cdot)$ is decreasing.

Imposing non-negative demand: We next consider the case where we formally restrict the additive demand function to be non-negative. Specifically, we show that the results of Proposition 6 for both the L and R rules still hold when we add a non-negativity demand constraint. We consider two ways of imposing such a constraint: (i) imposing a non-negativity constraint in the utility-maximization problem which will ensure that the final consumption is non-negative or (ii) taking the demand as the non-negative elements of the demanded quantities (i.e., a truncated demand function).

Recall that the additive demand function is given by: $d(\mathbf{p},\boldsymbol{\epsilon}) = d(\mathbf{p}) + \boldsymbol{\epsilon}$. A negative demand value is equivalent to the situation where the utility-maximization problem, $\max_{\mathbf{v}} u(\mathbf{p},\boldsymbol{\epsilon}) - \mathbf{p}^{\mathbf{T}}\mathbf{v}$, results in a $\mathbf{v}^*$ with a negative component. To avoid such a situation, a non-negative constraint can be added to the maximization problem that will become $\max_{\mathbf{0}\leq\mathbf{v}} u(\mathbf{p},\boldsymbol{\epsilon}) - \mathbf{p}^{\mathbf{T}}\mathbf{v}$. We denote $\mathbf{v}^{**}$ the optimal solution of this constrained problem. Since $\mathbf{v}^{**}$ is feasible to the unconstrained optimization problem and both optimization problems (i.e., with and without the non-negativity constraint) have the same objective function, then $CS(\mathbf{v}^*(\mathbf{p},\boldsymbol{\epsilon})) \geq CS(\mathbf{v}^{**}(\mathbf{p},\boldsymbol{\epsilon}))$ must hold. Thus, the results of Proposition 6 for the L and R rules continue to hold when we explicitly impose the non-negativity of the demand function.

Alternatively, if we truncate the demand function for each product i to be $z_i = \max\{v_i^*, 0\}$ (where $\mathbf{v}^*$ is the optimal solution of $\max_{\mathbf{v}} u(\mathbf{p},\boldsymbol{\epsilon}) - \mathbf{p}^{\mathbf{T}}\mathbf{v}$), the same argument applies, so that $CS(\mathbf{z}(\mathbf{p},\boldsymbol{\epsilon})) \geq CS(\mathbf{v}^{**}(\mathbf{p},\boldsymbol{\epsilon}))$ and, hence, Proposition 6 still holds.

- **H rule.** Consider a linear demand with an additive and symmetric noise. Thus, the inverse demand can be written as $d^{-1}(\mathbf{d},\boldsymbol{\epsilon}) = \mathbf{B}^{-1}(\overline{\mathbf{d}} + \boldsymbol{\epsilon} - \mathbf{d})$. For simplicity, we consider the case where $\mathbf{q}^{\mathbf{s}} = \overline{\mathbf{d}} - \mathbf{B}\mathbf{p}^{\mathbf{0}}$. Let $r^{\boldsymbol{\epsilon}}$ be a path consistent with the H rule so that $r^{\boldsymbol{\epsilon}}(w) = w\min\{d(\mathbf{p}^{\mathbf{0}},\boldsymbol{\epsilon}), \mathbf{q}^{\mathbf{s}}\}$ for $w \in [0,1]$. We then have

$$\begin{aligned}
\mathbb{E}[CS^H(\boldsymbol{\epsilon})] &= \mathbb{E}\left[\int_0^1 \sum_i (d_i^{-1}(w\min\{d(\mathbf{p}^{\mathbf{0}},\boldsymbol{\epsilon}),\mathbf{q}^{\mathbf{s}}\} - \boldsymbol{\epsilon}) - p_i^0)\min\{d_i(\mathbf{p}^{\mathbf{0}},\boldsymbol{\epsilon}), q_i\}dw\right] \\
&= \mathbb{E}\left[\int_0^1 \sum_i [\mathbf{B}^{-1}(\overline{\mathbf{d}} - \mathbf{B}\mathbf{p}^{\mathbf{0}} + \boldsymbol{\epsilon} - w\min\{d(\mathbf{p}^{\mathbf{0}},\boldsymbol{\epsilon}),\mathbf{q}^{\mathbf{s}}\}]_i \min\{d_i(\mathbf{p}^{\mathbf{0}},\boldsymbol{\epsilon}), q_i^s\}dw\right] \\
&= \mathbb{E}\left[\int_0^1 (\mathbf{q}^{\mathbf{s}} + \boldsymbol{\epsilon} - w\min\{d(\mathbf{p}^{\mathbf{0}},\boldsymbol{\epsilon}),\mathbf{q}^{\mathbf{s}}\})^T\mathbf{B}^{-1}\min\{d(\mathbf{p}^{\mathbf{0}},\boldsymbol{\epsilon}),\mathbf{q}^{\mathbf{s}}\}dw\right] \\
&= \mathbb{E}\left[(\mathbf{q}^{\mathbf{s}} + \boldsymbol{\epsilon})^T\mathbf{B}^{-1}\min\{\mathbf{q}^{\mathbf{s}} + \boldsymbol{\epsilon}, \mathbf{q}^{\mathbf{s}}\} - \frac{1}{2}\min\{\mathbf{q}^{\mathbf{s}} + \boldsymbol{\epsilon}, \mathbf{q}^{\mathbf{s}}\}^T\mathbf{B}^{-1}\min\{\mathbf{q}^{\mathbf{s}} + \boldsymbol{\epsilon}, \mathbf{q}^{\mathbf{s}}\}\right] \\
&= \frac{1}{2}(\mathbf{q}^{\mathbf{s}})^T\mathbf{B}^{-1}\mathbf{q}^{\mathbf{s}} + \mathbb{E}\left[\boldsymbol{\epsilon}^T\mathbf{B}^{-1}\min\{\boldsymbol{\epsilon},\mathbf{0}\} - \frac{1}{2}\min\{\boldsymbol{\epsilon},\mathbf{0}\}^T\mathbf{B}^{-1}\min\{\boldsymbol{\epsilon},\mathbf{0}\}\right].
\end{aligned}$$

Since we assume that the noises are independent and symmetric (in the sense of having a symmetric distribution centered at 0 for each product), we can assume without loss of generality that each ϵ_i takes the values A_i and $-A_i$ with probability 0.5. Therefore,

$$\begin{aligned}
\mathbb{E}[CS(\boldsymbol{\epsilon})^H] &= \frac{1}{2}(\mathbf{q}^{\mathbf{s}})^T\mathbf{B}^{-1}\mathbf{q}^{\mathbf{s}} + \mathbb{E}\left[\sum_{i,j}\min\{\epsilon_i,0\}[\mathbf{B}^{-1}]_{ij}\epsilon_j\right] - \mathbb{E}\left[\sum_{i,j}\min\{\epsilon_i,0\}\frac{[\mathbf{B}^{-1}]_{ij}}{2}\min\{\epsilon_j,0\}\right] \\
&= \frac{1}{2}(\mathbf{q}^{\mathbf{s}})^T\mathbf{B}^{-1}\mathbf{q}^{\mathbf{s}} + \mathbb{E}\left[\sum_{i}\min\{\epsilon_i,0\}[\mathbf{B}^{-1}]_{ii}\epsilon_i\right] - \mathbb{E}\left[\sum_{i\neq j}\min\{\epsilon_i,0\}\frac{[\mathbf{B}^{-1}]_{ij}}{2}\min\{\epsilon_j,0\}\right]
\end{aligned}$$

$$
\begin{aligned}
&-\mathbb{E}\left[\sum_i \min\{\epsilon_i,0\}\frac{[\mathbf{B^{-1}}]_{ii}}{2}\min\{\epsilon_i,0\}\right]\\
&=\frac{1}{2}(\mathbf{q^s})^T\mathbf{B^{-1}}\mathbf{q^s}+\frac{1}{2}\sum_i \min\{-A_i,0\}[\mathbf{B^{-1}}]_{ii}(-A_i)+\frac{1}{2}\cdot 0\\
&-\frac{1}{4}\sum_{i\neq j}\min\{-A_i,0\}\frac{[\mathbf{B^{-1}}]_{ij}}{2}\min\{-A_j,0\}-\frac{1}{2}\sum_i \min\{-A_i,0\}\frac{[\mathbf{B^{-1}}]_{ii}}{2}\min\{-A_i,0\}\\
&=\frac{1}{2}(\mathbf{q^s})^T\mathbf{B^{-1}}\mathbf{q^s}-\frac{1}{8}\mathbf{A^T}\mathbf{B^{-1}}\mathbf{A}+\frac{3}{8}\mathbf{A^T}\mathbf{D_{B^{-1}}}\mathbf{A},
\end{aligned}
$$

where $\mathbf{D_{B^{-1}}}$ is the diagonal matrix with the diagonal elements of $\mathbf{B^{-1}}$. Then, if $3\mathbf{D_{B^{-1}}}-\mathbf{B^{-1}}$ is positive semi-definite, then $\mathbb{E}[CS(\boldsymbol{\epsilon})^H]\geq \frac{1}{2}(\mathbf{q^s})^T\mathbf{B^{-1}}\mathbf{q^s}=CS_{det}$, hence concluding the proof.

Proof of Lemma 2

Under a multiplicative demand uncertainty, we know that each firm $i\in\{1,\ldots,n\}$ will solve a price-setting newsvendor problem. The optimal quantity, for a price vector $\mathbf{p}$, is given by $q_i(\mathbf{p})=d_i(\mathbf{p})F^{-1}(1-c_i/p_i)$, where F^{-1} is the inverse CDF of the noise ϵ. Then, the seller's optimization problem can be written as

$$
\max_{p_i}\; p_i d_i(\mathbf{p})\mathbb{E}\left[\min\left\{\epsilon, F^{-1}(1-c_i/p_i)\right\}\right]-c_i d_i(\mathbf{p})F^{-1}(1-c_i/p_i),
$$

where $d(\mathbf{p})=\overline{\mathbf{d}}-\mathbf{Bp}$. For ease of notation, we denote $\mathbf{D}\in\mathbb{R}^{n\times n}$ the diagonal matrix which elements match the ones of the diagonal of $\mathbf{B}$, $\mathbf{D_\Psi}\in\mathbb{R}^{n\times n}$ the diagonal matrix with $\mathbb{E}\left[\min\left\{\epsilon,F^{-1}(1-c_i/p_i)\right\}\right]$ in the i-th row-column component, and $\mathbf{D_\Theta}\in\mathbb{R}^{n\times n}$ the diagonal matrix with $F^{-1}(1-c_i/p_i)$ in the i-th row-column component. Note that the latter two matrices are functions of the price $\mathbf{p}$. By taking the first-order condition for each firm after replacing the linear demand function and moving to a vector notation, we have

$$
\mathbf{D_\Psi}(\overline{\mathbf{d}}-\mathbf{Bp})-\mathbf{BD_\Psi p}+\mathbf{BD_\Theta}=\mathbf{0}
$$

After some manipulation, we get that the equilibrium price vector $\mathbf{p^s}$ satisfies the following fixed-point equation:

$$
\mathbf{p^s}=\mathbf{c}+(\mathbf{B}+\mathbf{D})^{-1}(\overline{\mathbf{d}}-\mathbf{Bc})+(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{D_\Psi^{-1}D_\Theta}-\mathbf{I})\mathbf{Dc} \tag{16}
$$

By doing the same analysis for the deterministic version of the game, we obtain the equilibrium price $\mathbf{p^d}$ in closed-form expression to be given by:

$$
\mathbf{p^d}=\mathbf{c}+(\mathbf{B}+\mathbf{D})^{-1}(\overline{\mathbf{d}}-\mathbf{Bc}). \tag{17}
$$

Putting together Equations (16) and (17), we obtain

$$
\mathbf{p^s}=\mathbf{p^d}+(\mathbf{B}+\mathbf{D})^{-1}(\mathbf{D_\Psi^{-1}D_\Theta}-\mathbf{I})\mathbf{Dc}
$$

where $\mathbf{D_\Psi}$ and $\mathbf{D_\Theta}$ are functions of $\mathbf{p^s}$. Since $\mathbf{B}$ is an M-Matrix, so is $\mathbf{B}+\mathbf{D}$ and, hence, $(\mathbf{B}+\mathbf{D})^{-1}$ is non-negative. Also, note that $F^{-1}(1-c_i/p_i) \geq \mathbb{E}[\min\{\epsilon, F^{-1}(1-c_i/p_i)\}]$, then $\mathbf{D_\Psi^{-1}D_\Theta} - \mathbf{I} \geq \mathbf{0}$. One can see that $\mathbf{Dc} \geq \mathbf{0}$. Thus, we have $\mathbf{p^s} \geq \mathbf{p^d}$, which concludes the result.

Under an additive demand uncertainty, we know that each firm $i \in \{1,\dots,n\}$ will solve a price-setting newsvendor problem. The optimal quantity, for a price vector $\mathbf{p}$, is given by $q_i(\mathbf{p}) = d_i(\mathbf{p}) + F_i^{-1}(1-c_i/p_i)$, where F_i^{-1} is the inverse CDF of the noise ϵ_i. Then, the seller's optimization problem can be written as

$$\max_{p_i} p_i(d_i(\mathbf{p}) + \mathbb{E}\left[\min\left\{\epsilon, F_i^{-1}(1-c_i/p_i)\right\}\right]) - c_i(d_i(\mathbf{p}) + F_i^{-1}(1-c_i/p_i)),$$

where $d(\mathbf{p}) = \overline{\mathbf{d}} - \mathbf{Bp}$. For ease of notation, we denote $\mathbf{D} \in \mathbb{R}^{n\times n}$ the diagonal matrix which elements match the ones of the diagonal of $\mathbf{B}$, $\psi(\mathbf{p}): \mathbb{R}^n \to \mathbb{R}^n$ the function such that $\psi_i(\mathbf{p}) = \mathbb{E}\left[\min\left\{\epsilon_i, F_i^{-1}(1-c_i/p_i)\right\}\right]$. By taking first-order condition for each firm after replacing the linear demand function and moving to a vector notation, we have

$$\overline{\mathbf{d}} - \mathbf{Bp} - \mathbf{Bp} - \psi(\mathbf{p}) + \mathbf{Bc} = \mathbf{0}$$

After some manipulation, we get that the equilibrium price vector $\mathbf{p^s}$ satisfies the following fixed-point equation:

$$\mathbf{p^s} = \mathbf{c} + (\mathbf{B}+\mathbf{D})^{-1}(\overline{\mathbf{d}} - \mathbf{Bc}) + (\mathbf{B}+\mathbf{D})^{-1}\psi(\mathbf{p^s}) \tag{18}$$

The equilibrium price for the deterministic version was shown in Equation (17). Putting together Equations (18) and (17), we obtain

$$\mathbf{p^s} = \mathbf{p^d} + (\mathbf{B}+\mathbf{D})^{-1}\psi(\mathbf{p^s}).$$

Since $(\mathbf{B}+\mathbf{D})^{-1} \geq \mathbf{0}$ and $\psi(\mathbf{p}) \leq \mathbf{0}$ for any $\mathbf{p}$ (given that $\psi_i(\mathbf{p}) = \mathbb{E}\left[\min\left\{\epsilon_i, F_i^{-1}(1-c_i/p_i)\right\}\right] \leq \mathbb{E}[\epsilon_i] = 0$), then it must be that $\mathbf{p^s} \leq \mathbf{p^d}$, which concludes the proof.

Proof of Proposition 7

Before proving Proposition 7, we introduce the following lemma.

LEMMA 4. *Consider two price vectors $\mathbf{p^{(1)}}, \mathbf{p^{(2)}} \in \mathbb{R}^n$ such that $p_i^{(1)} \leq p_i^{(2)}$. Then, for any allocation rule $\mathcal{A}$ and any demand function $d(\mathbf{p}, \boldsymbol{\epsilon})$, we have*

$$CS^{\mathcal{A}}(\mathbf{p^{(1)}}, \boldsymbol{\epsilon}) \geq CS^{\mathcal{A}}(\mathbf{p^{(2)}}, \boldsymbol{\epsilon}).$$

Proof. The result follows directly from observing that the inverse demand function (that comes from the underlying utility function) is non-increasing, namely, $\frac{\partial d_i^{-1}}{\partial q_j} \leq 0$ (see Section 3 for more details). □

We next complete the proof of Proposition 7. We have

$$\begin{aligned}
\mathbb{E}[CS^H(\epsilon)] &= \mathbb{E}\left[\int_{\mathcal{C}^\epsilon}\sum_i (d_i^{-1}(\mathbf{r}^\epsilon,\boldsymbol{\epsilon}) - p_i^{sto})\mathbb{1}_{\{\mathbf{r}^\epsilon \le \min\{d(\mathbf{p^{sto}},\epsilon),\mathbf{q^{sto}}\}\}} dr_i^\epsilon\right] \\
&= \mathbb{E}\left[\int_{\mathcal{C}^\epsilon}\sum_i (d_i^{-1}\left(\frac{\mathbf{r}^\epsilon}{\epsilon}\right) - p_i^{sto})\mathbb{1}_{\{\mathbf{r}^\epsilon \le \min\{\epsilon d(\mathbf{p^0}),\mathbf{q^{sto}}\}\}} dr_i^\epsilon\right] \\
&= \mathbb{E}\left[\int_0^1 \sum_i \left(d_i^{-1}(w\min\{\epsilon d(\mathbf{p^{sto}}),\mathbf{q^{sto}}\}/\epsilon) - p_i^{sto}\right)\min\{\epsilon d_i(\mathbf{p^{sto}}), q_i^{sto}\} dw\right] \\
&= \mathbb{E}\left[\int_0^1 \sum_i \left(d_i^{-1}\left(wd(\mathbf{p^0})\right) - p_i^0\right) d_i(\mathbf{p^{sto}})\epsilon dw\right] \\
&= \int_0^1 \sum_i (d_i^{-1}(wd(\mathbf{p^{sto}})) - p_i^{sto}) d_i(\mathbf{p^{sto}}) dw \\
&\le \int_0^1 \sum_i (d_i^{-1}(wd(\mathbf{p^{det}})) - p_i^{sto}) d_i(\mathbf{p^{det}}) dw = CS_{det},
\end{aligned}$$

where the last inequality follows from Lemmas 2 and 4.

Proof of Proposition 8

Including the price dependence in the Consumer Surplus, we want to show that $\mathbb{E}[CS^H(\mathbf{p^s},\boldsymbol{\epsilon})] \le CS_{det}(\mathbf{p^d})$. In fact, this inequality follows from combining the results from Proposition 6, Lemma 4, and Lemma 2, namely, $\mathbb{E}[CS^H(\mathbf{p^s},\boldsymbol{\epsilon})] \le \mathbb{E}[CS^H(\mathbf{p^d},\boldsymbol{\epsilon})] \le CS_{det}(\mathbf{p^d})$.

Proof of Proposition 9

1. When $\delta = 0$, we have $B_{ij}^{-1} = 0$ and $B_{ii}^{-1} = \tilde{B}$. Therefore,

$$\mathbb{E}[CS^R(\boldsymbol{\epsilon})] = \frac{1}{2} n \left[\tilde{B}q^2 - 0.5A\tilde{B}(q - A)\right].$$

The worst-case values for the above expression is attained when $A = q/2$ (we have a quadratic function, so we simply use the first-order condition). By substituting $A = q/2$, we obtain

$$\frac{\mathbb{E}[CS^R(\boldsymbol{\epsilon})]}{CS_{det}} \ge \frac{7}{8}.$$

2. We next consider the case with $0 < \delta < \frac{b}{n-1}$. We can use the first-order condition to compute the value of A that yields the following worst-case ratio:

$$A_R^* = \frac{\tilde{B}q}{2B_{ii}^{-1}}.$$

Recall that $A \le q$. As a result, we need to separate the analysis into two cases.

When $\frac{\tilde{B}}{B_{ii}^{-1}} \le 2$, the ratio attains its worst value when $A = A_R^*$. In this case, we have

$$\frac{\mathbb{E}[CS^R(\boldsymbol{\epsilon})]}{CS_{det}} \ge 1 - \frac{1}{8}\frac{\tilde{B}}{B_{ii}^{-1}} \ge \frac{3}{4}.$$

When $\frac{\tilde{B}}{B_{ii}^{-1}} > 2$, the ratio attains its worst value when $A = q$ (in this case, the derivative is always positive). We then have

$$\frac{\mathbb{E}[CS^R(\boldsymbol{\epsilon})]}{CS_{det}} \geq 1 - \frac{1}{2}(1 - \frac{B_{ii}^{-1}}{\tilde{B}}) = \frac{1}{2} + \frac{B_{ii}^{-1}}{2\tilde{B}}.$$

Note that by definition $\tilde{B} = B_{ii}^{-1} + (n-1)B_{ij}^{-1}$. Therefore,

$$\frac{\tilde{B}}{B_{ii}^{-1}} = 1 + (n-1)\frac{B_{ij}^{-1}}{B_{ii}^{-1}} \leq n.$$

For any value of n, we then have the following:

$$\frac{\mathbb{E}[CS^R(\boldsymbol{\epsilon})]}{CS_{det}} \geq \frac{1}{2} + \frac{1}{2n}. \quad \square$$

Proof of Proposition 10

Consider the H allocation rule with two different paths: (i) $\mathcal{C}_1^{\delta}$ from $\mathbf{0}$ to $\min\{d(\mathbf{p}), s(\mathbf{p}, \boldsymbol{\delta})\}$ with the parametric function $\mathbf{r}^{\boldsymbol{\delta}}$ and (ii) $\mathcal{C}_2^{\boldsymbol{\delta}}$ from $\min\{d(\mathbf{p}), s(\mathbf{p}, \boldsymbol{\delta})\}$ to $d(\mathbf{p})$ with the parametric function $\mathbf{t}^{\boldsymbol{\delta}}$. We let $\mathcal{C}$ be the union of these two paths which goes from $\mathbf{0}$ to $d(\mathbf{p})$ with a parametric function denoted by $\mathbf{u}$. We then have

$$\begin{aligned}
\mathbb{E}[CS^H(\boldsymbol{\delta})] &= \mathbb{E}\left[\int_{\mathcal{C}_1^{\boldsymbol{\delta}}} \sum_i (d_i^{-1}(\mathbf{r}^{\boldsymbol{\delta}}) - p_i^0)\mathbb{1}_{\{\mathbf{r}^{\boldsymbol{\delta}} \leq \min\{d(\mathbf{p^0}), s(\mathbf{p^0}, \boldsymbol{\delta})\}\}} dr_i^{\boldsymbol{\delta}}\right] \\
&= \mathbb{E}\left[\int_{\mathcal{C}_1^{\boldsymbol{\delta}}} \sum_i (d_i^{-1}(\mathbf{r}^{\boldsymbol{\delta}}) - p_i^0) dr_i^{\boldsymbol{\delta}}\right] \\
&\leq \mathbb{E}\left[\int_{\mathcal{C}_1^{\boldsymbol{\delta}}} \sum_i (d_i^{-1}(\mathbf{r}^{\boldsymbol{\delta}}) - p_i^0) dr_i^{\boldsymbol{\delta}}\right] + \mathbb{E}\left[\int_{\mathcal{C}_2^{\boldsymbol{\delta}}} \sum_i (d_i^{-1}(\mathbf{t}^{\boldsymbol{\delta}}) - p_i^0) dt_i^{\epsilon}\right] \\
&= \mathbb{E}\left[\int_{\mathcal{C}} \sum_i (d_i^{-1}(\mathbf{u}) - p_i^0) du_i\right] = CS_{det}.
\end{aligned}$$

Proof of Proposition 11

When both demand and supply are stochastic, the path under the H rule goes from $\mathbf{0}$ to $\min\{\mathbf{s}, \mathbf{d}\}$ with a weight of 1 and from $\min\{\mathbf{s}, \mathbf{d}\}$ to $\mathbf{d}$ with a weight of 0. Recall that in this case, $s_i = \delta_i d_i(\mathbf{p})$. We denote $\mathbf{r}^{\epsilon,\boldsymbol{\delta}}$ the parametric function of this allocation rule.

Consider any path consistent with the H rule and $\mathbf{r}^{\epsilon,\boldsymbol{\delta}}$ its parametric function. We then have

$$\begin{aligned}
\mathbb{E}_{\epsilon,\boldsymbol{\delta}}[CS^H(\epsilon, \boldsymbol{\delta})] &= \mathbb{E}_{\epsilon,\boldsymbol{\delta}}\left[\int_{\mathcal{C}^{\epsilon,\boldsymbol{\delta}}} \sum_i (d_i^{-1}(\mathbf{r}^{\epsilon,\boldsymbol{\delta}}, \epsilon) - p_i^0)\mathbb{1}_{\{\mathbf{r}^{\epsilon,\boldsymbol{\delta}} \leq \min\{d(\mathbf{p^0}, \epsilon), \mathbf{s}\}\}} dr_i^{\epsilon,\boldsymbol{\delta}}\right] \\
&= \mathbb{E}_{\epsilon,\boldsymbol{\delta}}\left[\int_{\mathcal{C}^{\epsilon,\boldsymbol{\delta}}} \sum_i (d_i^{-1}\left(\frac{\mathbf{r}^{\epsilon,\boldsymbol{\delta}}}{\epsilon}\right) - p_i^0)\mathbb{1}_{\{\mathbf{r}^{\epsilon,\boldsymbol{\delta}} \leq \min\{d(\mathbf{p^0}, \epsilon), \mathbf{s}\}\}} dr_i^{\epsilon,\boldsymbol{\delta}}\right] \\
&= \mathbb{E}_{\epsilon,\boldsymbol{\delta}}\left[\int_0^1 \sum_i \left(d_i^{-1}(w \min\{\epsilon d(\mathbf{p^0}), \mathbf{D}_{\boldsymbol{\delta}} d(\mathbf{p^0})\}/\epsilon) - p_i^0\right) \min\{\epsilon d_i(\mathbf{p^0}), \delta_i d_i(\mathbf{p^0})\} dw\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_{\epsilon,\boldsymbol{\delta}}\left[\int_0^1 \sum_i \left(d_i^{-1}\left(wd(\mathbf{p^0})\min\left\{1,\frac{\delta_i}{\epsilon}\right\}\right)-p_i^0\right)d_i(\mathbf{p^0})\epsilon\min\{1,\delta_i/\epsilon\}\,dw\right]\\
&= \mathbb{E}_{\epsilon,\boldsymbol{\delta}}\left[\sum_i \int_0^{\min\{1,\delta_i/\epsilon\}} (d_i^{-1}(vd(\mathbf{p^0}))-p_i^0)d_i(\mathbf{p^0})\epsilon dv\right]\\
&\le \mathbb{E}_{\epsilon}\left[\int_0^{\min\{1,1/\epsilon\}} \sum_i (d_i^{-1}(vd(\mathbf{p^0}))-p_i^0)d_i(\mathbf{p^0})\epsilon dv\right]\\
&\le \int_0^1 \sum_i (d_i^{-1}(vd(\mathbf{p^0}))-p_i^0)d_i(\mathbf{p^0})dv = CS_{det},
\end{aligned}
$$

where the first and second inequalities follow from Jensen's inequality. The matrix $\mathbf{D}_{\boldsymbol{\delta}}$ is a diagonal matrix with δ_i in the i-th row-column.

Proof of Proposition 12

- R rule. In this case, the path goes from $\mathbf{0}$ to $d(\mathbf{p},\boldsymbol{\epsilon})$ with an allocation for each product i such that $\min\{1,\frac{s_i(\mathbf{p^0},\boldsymbol{\delta})}{d_i(\mathbf{p^0},\boldsymbol{\epsilon})}\} = \min\{1,\frac{d_i(\mathbf{p})+\delta_i}{d_i(\mathbf{p})+\epsilon_i}\}$. We have

$$
\begin{aligned}
\mathbb{E}_{\boldsymbol{\epsilon},\boldsymbol{\delta}}[CS^R(\boldsymbol{\epsilon},\boldsymbol{\delta})] &= \mathbb{E}_{\boldsymbol{\epsilon},\boldsymbol{\delta}}\left[\int_{\mathcal{C}^{\boldsymbol{\epsilon}}} \sum_{i=1}^n (d_i^{-1}(r^{\boldsymbol{\epsilon}},\boldsymbol{\epsilon})-p_i^0)\min\left\{1,\frac{s_i(\mathbf{p},\boldsymbol{\delta})}{d_i(\mathbf{p^0},\boldsymbol{\epsilon})}\right\}dr_i^{\boldsymbol{\epsilon}}\right]\\
&= \mathbb{E}_{\boldsymbol{\epsilon},\boldsymbol{\delta}}\left[\int_{\mathcal{C}^{\boldsymbol{\epsilon}}} \sum_{i=1}^n (d_i^{-1}(r^{\boldsymbol{\epsilon}}-\boldsymbol{\epsilon})-p_i^0)\min\left\{1,\frac{d_i(\mathbf{p^0})+\delta_i}{d_i(\mathbf{p^0})+\epsilon_i}\right\}dr_i^{\boldsymbol{\epsilon}}\right]\\
&\le \mathbb{E}_{\boldsymbol{\epsilon}}\left[\int_{\mathcal{C}^{\boldsymbol{\epsilon}}} \sum_{i=1}^n (d_i^{-1}(r^{\boldsymbol{\epsilon}}-\boldsymbol{\epsilon})-p_i^0)\min\left\{1,\frac{d_i(\mathbf{p^0})}{d_i(\mathbf{p^0})+\epsilon_i}\right\}dr_i^{\boldsymbol{\epsilon}}\right] \le CS_{det},
\end{aligned}
$$

  where the first inequality follows from $\mathbb{E}[\min\{X,Y\}] \le \min\{\mathbb{E}[X],\mathbb{E}[Y]\}$. The second inequality follows a similar argument as in the proof for the case with deterministic supply.

- L rule. The result that if d_i^{-1} is concave, then $\mathbb{E}_{\epsilon,\delta}[CS^L] \le CS^{det}$ follows directly from the result of the R rule. As observed in the single-product setting, the inequality does not hold for any demand function.

Constrained Utility Maximization Problem

In this section, we consider an alternative framework to capture the substitution behavior of consumers. The motivation of this utility modeling framework is to capture the demand substitution behavior in the utility-maximization problem.

Alternative Utility Model

We consider a utility function $u:\mathbb{R}^n \times \Omega \to \mathbb{R}$, where Ω is the support of the random terms. Then, given a price vector $\mathbf{p}$ and a vector of available supply $\mathbf{q} \in \mathbb{R}^n \cup \{\infty\}^n$, the utility-maximization problem solved by a representative consumer can be formulated as

$$
\max_{\mathbf{0}\le\mathbf{v}\le\mathbf{q}} u(\mathbf{v},\boldsymbol{\epsilon}) - \mathbf{p^T v}. \tag{19}
$$

The optimal solution of problem (19) will be denoted as the *constrained* demand function $y(\mathbf{p}, \boldsymbol{\epsilon})$. The prefix *constrained* is used to differentiate this setting from the traditional demand function which will be reserved for the unconstrained solution of the same maximization problem. We use d to denote the traditional demand function and d^{-1} its inverse (note that $d^{-1} = \nabla_{\mathbf{v}} u$). We make this distinction to examine the role of uncertainty under both settings. In particular, the nature of an additive or a multiplicative noise will be considered with respect to the demand function d. Indeed, one can see that the constrained demand function y will not satisfy additive or multiplicative effects when facing scarcity. It is important to observe that by construction, y is upper bounded by the available supply. Thus, in this modeling framework there is no need to incorporate allocation rules for the available supply.

Then, the Consumer Surplus under this utility model can be expressed as $u(y(\mathbf{p}, \boldsymbol{\epsilon}), \boldsymbol{\epsilon}) - \mathbf{p}^{\mathbf{T}} y(\mathbf{p}, \boldsymbol{\epsilon})$ or in terms of the inverse demand function as $\int_{\mathcal{C}^{\boldsymbol{\epsilon}}} (\nabla_{\mathbf{v}} u(\mathbf{r}^{\boldsymbol{\epsilon}}, \boldsymbol{\epsilon}) - \mathbf{p})^T \mathbf{d}\mathbf{r}^{\boldsymbol{\epsilon}} = \int_{\mathcal{C}^{\boldsymbol{\epsilon}}} \sum_i (d_i^{-1}(\mathbf{r}^{\boldsymbol{\epsilon}}, \boldsymbol{\epsilon}) - p_i) dr_i^{\boldsymbol{\epsilon}}$, where $\mathcal{C}^{\boldsymbol{\epsilon}}$ is any parametrized path from $\mathbf{0}$ to $y(\mathbf{p}, \boldsymbol{\epsilon})$.

Impact of Demand Uncertainty

For a setting with a single product, the constrained utility function is equivalent to the unconstrained setting under with the H allocation rule. Thus, the same results hold with respect to the comparison of $\mathbb{E}[CS(\epsilon)]$ and CS^{det}. Namely, $\mathbb{E}[CS(\epsilon)] \leq CS^{det}$ for a multiplicative noise and $\mathbb{E}[CS(\epsilon)] \geq CS^{det}$ for an additive noise when d^{-1} is convex. However, this equivalence does not hold for the setting with multiple products. Instead, we re-derive the results for this alternative model in Propositions 13 and 14 below. We let $CS(\epsilon)$ be the Consumer Surplus under the constrained utility model under consideration. As before, we let $\mathbf{q} = d(\mathbf{p})$.

Proposition 13. For the setting with multiple products and a multiplicative noise, we have

$$\mathbb{E}[CS(\epsilon)] \leq CS^{det}.$$

We next consider the case with an additive noise.

Proposition 14. Consider the setting with multiple products and an additive noise. Then, under the same conditions as Proposition 6 for the H rule, we have

$$\mathbb{E}[CS(\boldsymbol{\epsilon})] \geq CS^{det}.$$

As discussed, the constrained utility-maximization problem does not require allocation rules, since the maximization problem incorporates an inventory constraint to ensure that demand does not exceed supply. The drawback is that there may be cases where we end up overstating the Consumer Surplus. For example, consider a demand with a multiplicative noise for which a positive demand shock will double the demand value (it is equivalent to duplicating each infinitesimal consumer). Consider the following concrete example with a single product and a multiplicative demand function $d(p, \epsilon) = \epsilon(1 - p)$.[15] We let $q = p = 0.5$. In the case with no uncertainty (i.e., $\epsilon = 1$), the Consumer Surplus will be equal to $0.5 \times 0.5/2 = 1/8$ (see the left panel of Figure 7 for an illustration). If we add a positive demand shock such that each infinitesimal consumer is duplicated (i.e., $\epsilon = 2$), then the Consumer Surplus under the constrained utility model is equal to $[0.5 \times (0.5 + 0.25)/2]/2 = 3/16$ (see the right panel of Figure 7). Thus, the Consumer Surplus obtained under the positive demand shock is larger relative to the deterministic case. However, if each infinitesimal consumer has the same chance of receiving the item, it is natural to expect that the Consumer Surplus will remain unchanged for any value of $\epsilon \geq 1$. In fact, the latter property is satisfied for the R rule as shown in Section 2. Specifically, the value of the Consumer Surplus in the previous example under the R rule is equal to $1/8$ for any $\epsilon \geq 1$.

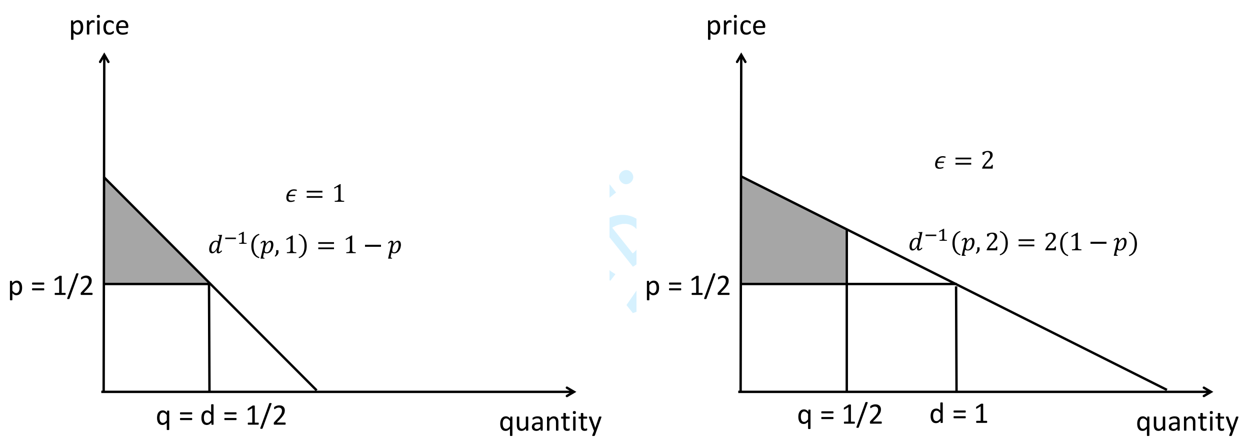

Figure 7 The shaded region denotes the Consumer Surplus according to the constrained utility-maximization problem when $\epsilon = 1$ (left panel) and $\epsilon = 2$ (right panel).

Finally, we compare the Consumer Surplus under the constrained utility-maximization problem relative to the Consumer Surplus based on the framework from Sections 2 and 3 under the H rule. Figure 8 depicts the case with $n = 2$, where the first product has its supply below demand, whereas the second product has infinite supply. The point (d_1, d_2) is the unconstrained solution of the utility-maximization problem. This solution cannot materialize in practice due to the scarcity on the first product. In the case of the H rule (see Section 3), we can compute the Consumer Surplus by evaluating the utility function at $\min\{\mathbf{d}, \mathbf{q}\}$ minus the sum-product of the prices and quantities. In the case of the constrained utility-maximization problem, the solution results in the point $\mathbf{y}$, which will substitute some units of the first product by increasing the demand for the

[15] In this case, the utility function must be given by $u(v, \epsilon) = v - v^2/(2\epsilon) - pv$.

second product. Then, the Consumer Surplus in this case is the utility evaluated at this point minus the payment. It can thus be clearly seen that the latter Consumer Surplus is larger than the value under the H rule (a formal proof of this inequality is provided in Lemma 5 below).

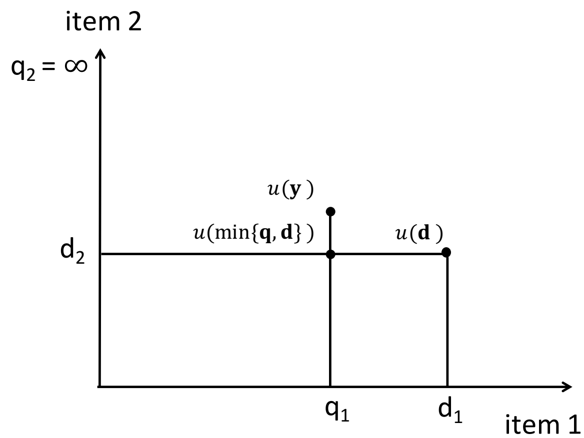

Figure 8 Illustration of the solutions under the constrained and unconstrained settings for the H rule.

Proof of Proposition 13

We will show that $CS(\epsilon) = CS(\epsilon)^H$, where $CS(\epsilon)^H$ is the Consumer Surplus from the traditional unconstrained utility model under the H rule. Note that if $\epsilon < 1$, then $y(\mathbf{p},\epsilon) = d(\mathbf{p})\epsilon = \epsilon\mathbf{q}$, whereas under the H rule the consumption is $\min\{d(\mathbf{p},\epsilon),\mathbf{q}\} = \min\{\epsilon d(\mathbf{p}),\mathbf{q}\} = \min\{\epsilon\mathbf{q},\mathbf{q}\} = \epsilon\mathbf{q}$ and, hence, $CS(\epsilon) = CS(\epsilon)^H$. If $\epsilon > 1$, then in the case of $CS(\epsilon)$, the solution of the consumer utility-maximization problem will lead to $d(\mathbf{p}) = \mathbf{q}$. Similarly, the consumption under the H rule must also satisfy $\epsilon\min\{d(\mathbf{p}),\mathbf{q}\} = \min\{\epsilon\mathbf{q},\mathbf{q}\} = \mathbf{q}$. Therefore, in both cases the Consumer Surplus is $u(\mathbf{q}) - \mathbf{p}^{\mathbf{T}}\mathbf{q}$ and, thus, $CS(\epsilon) = CS^H(\epsilon)$. The desired result then follows from using the inequality between $\mathbb{E}[CS(\epsilon)^H]$ and CS^{det}. □

Proof of Proposition 14

The proof relies on the following lemma.

LEMMA 5. *Let $CS(\boldsymbol{\epsilon})$ be the Consumer Surplus under the constrained utility model and $CS(\boldsymbol{\epsilon})^H$ under the traditional (unconstrained) model with the H rule. Then, the following holds:*

$$\mathbb{E}[CS(\boldsymbol{\epsilon})] \geq \mathbb{E}[CS^H(\boldsymbol{\epsilon})].$$

Proof. For any noise realization $\boldsymbol{\epsilon}$, we let $\mathbf{s} = \min\{d(\mathbf{p},\boldsymbol{\epsilon}),\mathbf{q}\}$. We have $CS(\boldsymbol{\epsilon}) = u(y(\mathbf{p},\boldsymbol{\epsilon}),\boldsymbol{\epsilon}) - \mathbf{p}^{\mathbf{T}}y(\mathbf{p},\boldsymbol{\epsilon}) \geq u(\mathbf{s},\boldsymbol{\epsilon}) - \mathbf{p}^{\mathbf{T}}\mathbf{s} = CS^H(\boldsymbol{\epsilon})$ because $\mathbf{s}$ is feasible to the constrained utility-maximization problem. Taking the expectation on both sides concludes the proof. □

The result of Proposition 14 follows directly from Lemma 5 and from the inequality between $\mathbb{E}[CS(\boldsymbol{\epsilon})^H]$ and CS^{det}. □