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Consumer Surplus Under Demand Uncertainty

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Consumer Surplus is traditionally defined for the case where demand is a deterministic function of the price. However, demand is usually stochastic and hence stock-outs can occur. Policy makers who consider the impact of different regulations on Consumer Surplus often ignore the impact of demand uncertainty. We present a definition of the Consumer Surplus under stochastic demand. We then use this definition to study the impact of demand and supply uncertainty on consumers in several cases (additive and multiplicative demand noise). We show that, in many cases, demand uncertainty hurts consumers. We also derive analytical bounds on the ratio of the Consumer Surplus relative to the deterministic setting under linear demand. Our results suggest that ignoring uncertainty may severely impact the Consumer Surplus value.

Key words: consumer surplus; demand uncertainty; OM-economics interface; rationing capacity
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1. Introduction

Customer welfare is often measured by using a metric called the Consumer Surplus. This metric was proposed in 1844 by Jules Dupuit who defined the Consumer Surplus as the difference between consumers' willingness to pay and market price. Subsequently, this concept was extensively studied in the economics literature and applied to a multitude of domains. In these applications, however, it is assumed that the demand function is deterministic and that the products are always available (i.e., no stock-outs).

In real-world settings, demand is stochastic by nature. Data scientists and statisticians constantly aim to improve demand prediction algorithms, but no method will consistently yield a perfect prediction. Consequently, firms will do their best to predict demand but prediction errors remain inevitable. The operations management (OM) community has proposed to study such settings by considering a stochastic demand function and intends to make optimal decisions under uncertainty. OM researchers have studied various problems such as supply chain management, revenue management, and queuing systems. While the focus has mainly been on the firm's perspective, several recent research trends (e.g., sustainable operations) study the problem from the consumers' perspective. In particular, policy makers and regulators are often interested in assessing the impact of specific policies or interventions on consumers. In this context, the relevant question for a policy maker is whether the standard (presumably deterministic) approach to calculate the Consumer Surplus ends up overstating its value and, hence, paint an inaccurate picture. To measure customer welfare in such settings, one needs a general definition of Consumer Surplus under demand uncertainty, while accounting for the events where products are not available (i.e., stock-outs).

As observed in Krishnan (2010), demand uncertainty renders the welfare analysis more intricate. Including a stochastic term in the demand function may hinder the existence of an underlying representative customer utility function and, thus, utility analysis may not be possible. Besides this concern, demand uncertainty may also lead to scarcity and stock-outs. Given a price, the demand function represents the maximum number of units to be consumed (or at least desired to be consumed). Such a demand function captures multiple consumers, each endorsed with a particular willingness to pay for the product. When demand exceeds supply, not all customers who are willing to buy the product will be served.[1] As a result, the Consumer Surplus will depend on the set of

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consumers who receive the product. We formalize this intuition by using the notion of capacity allocation rule (see Section 2.2.1).

Several prior papers have studied the impact of demand uncertainty but their focus was mainly on prices, production quantities, and firms' profits. A classical example is the newsvendor problem in which the optimal production quantity is derived when the firm only knows the demand distribution (as opposed to the exact realization). Other studies, such as Mills (1959) and Karlin and Carr (1962), compared the optimal prices and profits earned by firms in the stochastic setting relative to their deterministic counterparts for additive and multiplicative demand noises. However, these studies do not address the impact of demand uncertainty on consumers. As mentioned, assessing the impact of policies or interventions on consumers is of great interest to policy makers. Given the stochastic nature of most real-world settings, one needs a precise way to measure customer welfare when demand and supply are uncertain. This paper aims to bridge this gap by proposing a first step in answering this question.

What happens if we ignore stock-outs and demand uncertainty when computing the Consumer Surplus? Obviously, the answer will depend on which customers receive the limited supply. In this paper, we present a formal definition of the Consumer Surplus under stochastic demand and stochastic supply. We then compare the Consumer Surplus relative to the deterministic setting for various noise structures and capacity allocation rules. Our model and analyses allow us to answer the question of whether more uncertainty is good or bad for consumers—and to what extent. It can thus provide guidelines to firms and regulators on the potential need for investing efforts in collecting additional data and in developing more sophisticated prediction methods to reduce demand uncertainty.

1.1. Contributions

In economics, the concept of Consumer Surplus has mainly focused on cases where demand is a deterministic function of the price, so that products are always available (see, e.g., Tirole 1988, Vives 2001). As mentioned, many real-world settings are stochastic by nature and modeled via a stochastic demand function. With the goal of accurately measuring customer welfare, this paper offers a first step in generalizing the notion of Consumer Surplus and examining the impact of demand and supply uncertainty on consumers. We next summarize our contributions.

- *Presenting a generalization of the Consumer Surplus notion*. As discussed, when products are not necessarily available, the Consumer Surplus will naturally depend on which customers receive the limited supply. We use capacity allocation rules to model the way available units are allocated to consumers when demand exceeds supply. We first provide a rigorous definition of a capacity allocation rule. We then present a general definition of the Consumer Surplus for multiple products under stochastic demand. To our knowledge, this paper is the first to propose such a general extension.
- *Studying the impact of demand and supply uncertainty on consumers*. Armed with our Consumer Surplus definition, we study the impact of demand and supply uncertainty on consumers for several special cases (multiplicative and additive demand noises). Specifically, we compare the expected Consumer Surplus to the deterministic setting, under the same prices. We also extend our results to the setting where prices are endogenously determined. We show that in many cases, demand uncertainty hurts consumers. Under a demand with multiplicative noise, consumers are always better off in the deterministic setting. Interestingly, this result holds for any demand function, any noise distribution, and any allocation rule. Under an additive demand noise, we show that the impact of demand uncertainty depends on the allocation rule and on the convexity properties of the demand. Finally, we show that regardless of the type of noise, the expected Consumer Surplus under stochastic supply is always lower relative to its deterministic counterpart. Ultimately, our results suggest that ignoring uncertainty may severely impact the Consumer Surplus.
- *Deriving analytical bounds on the Consumer Surplus ratio relative to the deterministic setting under linear demand*. Under additional assumptions, we derive bounds on the impact of demand uncertainty on the Consumer Surplus. We show that the expected Consumer Surplus under stochastic demand can be as far as 50% relative to the deterministic setting.

1.2. Literature Review

The Consumer Surplus is typically defined as the difference between consumers' willingness to pay and market price. This definition was first introduced by Jules Dupuit in 1844 as *utilité relative*. Later on, Alfred Marshall relabeled this concept as Consumer Surplus in *the Principles of Economics* in 1890, denoting it as the upper triangle of the inverse demand curve. Since then, hundreds of academic papers were published

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on this topic (we cannot give proper credit to the extensive research conducted on this topic). Several studies have developed analytical frameworks (see, e.g., Takayama 1982, Willig 1976), whereas others have focused on applications spanning a multitude of contexts.

The basic definition of the Consumer Surplus relies on computing the area between the market price p and the inverse demand curve (see, e.g., Pindyck and Rubinfeld 2018, Tirole 1988, as well as section 2.1 below). This definition rests on the assumption that the product is always available and that there are no stock-outs. Specifically, in most previous work, the framework developed to compute the Consumer Surplus focuses on the case where the demand curve is a deterministic function of the price. As mentioned, many real-world settings are modeled using a stochastic demand. To measure customer welfare in such settings, one needs an appropriate definition of the Consumer Surplus. Indeed, calculating the Consumer Surplus while ignoring stock-outs may lead to a severe overestimate. Furthermore, the correct estimate naturally depends on the set of customers who receive the limited supply. Extending the notion of Consumer Surplus under stochastic demand and supply is one of the main goals of this paper.

Several papers on peak load pricing and capacity investments by a power utility under stochastic demand address partially this modeling issue (see Brown and Johnson 1969, Crew et al. 1995). Nevertheless, the models developed in this literature are not applicable to settings where consumers arrive randomly and not according to their valuations. Specifically, in Brown and Johnson (1969), the authors assume that the utility power facility has access to the willingness to pay of consumers, so that it can decline the ones with lowest valuations. This assumption is not justifiable in a setting where a first-come-first-serve logic with random arrivals is more suitable (e.g., the retail industry). In a similar spirit, Ha (1997) and Liu and Van Ryzin (2008) study optimal rationing policies for a firm that faces uncertain demand. In Raz and Ovchinnikov (2015), the authors study a price-setting newsvendor model for public goods and consider the Consumer Surplus for a single product with linear additive stochastic demand. In Cohen et al. (2015), the authors study the impact of demand uncertainty on consumer subsidies for green technology adoption and propose a special case of the definition presented in this paper. To our knowledge, this paper is the first to provide a rigorous extension of the Consumer Surplus definition under stochastic demand for multiple products and any capacity allocation rule.

The second contribution of this paper is to use our Consumer Surplus definition to study the impact of demand and supply uncertainty on consumers. Examining the impact of demand uncertainty (or stock-outs) has been a common research topic in the OM literature. Examples include inventory and supply chain (Gupta and Maranas 2003, Kaya and Özer 2011, Özer and Wei 2004), capacity investment (Goyal and Netessine 2007), and subsidies for green technology adoption (Cohen et al. 2015). The results of this paper can be of interest to policy makers when assessing the customer welfare in various settings where uncertainty is inherent.

Structure of the paper. In section 2, we present our results for the single product model. We then extend the treatment for multiple products in section 3. In section 4, we consider the impact of supply uncertainty. Finally, our conclusions are reported in section 5. Most of the proofs of the technical results are relegated to the appendix.

2. Single Product

In this section, we consider the case of a supplier (also referred to as seller) selling a single product. The single-product setting is an important building block as it allows us to develop the intuitions and tools needed for the setting with multiple products considered in section 3. We assume that the seller is facing a stochastic demand curve $d(p, \epsilon)$. Specifically, we explore two commonly-used noise dependences: (i) additive, that is, $d(p, \epsilon) = d(p)+\epsilon$, where ϵ is a random variable with mean 0; and (ii) multiplicative, that is, $d(p, \epsilon) = \epsilon d(p)$, where ϵ is a positive random variable with mean 1.[2] The function $d(p)$ is called the *nominal demand function* and corresponds to the setting with deterministic demand (i.e., $\epsilon = 0$ with probability 1 in the case with additive noise and $\epsilon = 1$ with probability 1 for the multiplicative noise). We assume that $d(p)$ is strictly decreasing. We refer to $d^{-1}(q)$ as the *nominal inverse demand*. We assume that $d^{-1}(q,\epsilon)$ exists and is defined so that $d(d^{-1}(q,\epsilon),\epsilon) = q$.

As is common in the OM literature, we assume that the nominal demand $d(p)$ is only a function of the price, omitting arguments such as income and utility (which are explicit arguments of the Marshallian and Hicksian demands, respectively).[3] If $d(p)$ comes from a representative consumer solving the utility-maximization problem (UMP), then this utility function has to be quasilinear (so that the welfare can be properly measured), namely, the representative consumer utility function is such that $U(x_0, x_1) = x_0 + u(x_1)$, where x_0 is the consumption of a numeraire good with price equal to 1 (for more details, see Varian 1992, Chapter 10). The UMP can be written as $\max_{x_0,x_1} x_0 + u(x_1)$ subject to the budget constraint $x_0 + px_1 \leq I$. Equivalently, one can write $I + \max_{x_1} u(x_1) - px_1$, resulting in a demand that only depends on the price p. Alternatively, one can consider a representative consumer solving the expenditure-minimization problem (EMP) given by $\min_{x_0,x_1} x_0 + px_1$ subject to $U(x_0, x_1) \geq u'$. The latter can be rewritten as $u' + \min_{x_1} px_1 - u(x_1)$. As discussed in Varian (1992), the income is assumed to be high enough so that the product consumption is independent of income. We thus assume throughout the paper that $d(p)$ (and $d(p, \epsilon)$ in the stochastic case) is only a function of the price (and noise). Assuming that demand integrability conditions are satisfied, this is equivalent to require a quasilinear utility function and that the representative consumer's income is high enough. Our goal is to study the Consumer Surplus under a general stochastic demand function. First, we briefly recall the definition under deterministic demand.

2.1. Deterministic Demand

Consumer Surplus is an economic measure of consumer net welfare to quantify product consumption given the incurred expenditures. This can be computed as the area between the market price p and the inverse demand curve (see an illustration in Figure 1). For given values of p and $q = d(p)$, the Consumer Surplus under deterministic demand, CS_{det}, can be computed as:[4]

$$CS_{det} = \int_0^{d(p)} [d^{-1}(w) - p]dw. \qquad (1)$$

Equivalently, the Consumer Surplus can be computed by integrating over the price space:

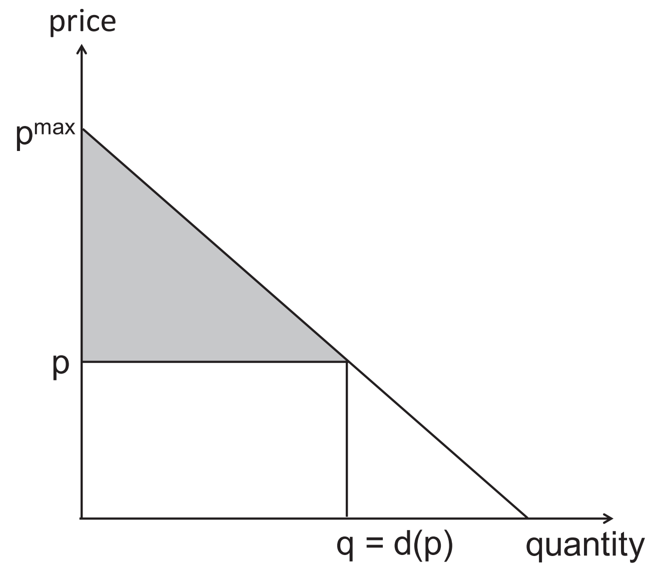
$$CS_{det} = \int_p^{p^{max}} d(z)dz,$$

where p^{max} is $d^{-1}(0)$, that is, the inverse demand at zero or $+\infty$ when $d^{-1}(0)$ is not defined (e.g., for $d(p) = 1/p$); sometimes called "the null price." In the remainder of this paper, we focus on the Consumer Surplus expression in Equation (1), where the integration is taken over the quantity space. Note that since $d(z) \geq 0$ and $p \leq d^{-1}(0)$, one can see that CS_{det} is non-increasing in p.

2.2. Stochastic Demand

When demand is stochastic, defining the Consumer Surplus is more subtle due to the possibility of stock-outs. More precisely, if the demand at price p under a particular realization ϵ happens to be greater than the available supply q (i.e., $d(p,\epsilon) - q > 0$), some consumers who are willing to make a purchase will not be served due to the limited supply. This stock-out event clearly affects customer welfare and, hence, should be accounted for in the Consumer Surplus definition. To

Figure 1 Illustration of the Consumer Surplus for a Single Product Under a Deterministic Linear Demand



further motivate this issue, consider the following "discretized" example with two customers with valuations 8 and 6, and assume that there are two items available at price $p = 3$. In this case, both items are allocated, and the Consumer Surplus is $CS = (8-3)+(6-3) = 8$. We now introduce a positive shock that doubles the demand while keeping the same proportion of customers at each valuation (such a shock corresponds to a multiplicative demand noise with $\epsilon = 2$). What is the Consumer Surplus in this case? It is clear that the answer will depend on how both available items are allocated to the four customers. Our goal is to propose a method for computing customer welfare under any possible allocation. In the previous example, one can interpret the four customers' valuations as the marginal utility of a representative consumer who maximizes welfare subject to a maximum consumption of two items. In this case, the two customers with the highest valuation will receive the item. However, from a modeling perspective, this might be restrictive since it does not provide any flexibility on the allocation. For example, what happens if the items are randomly allocated to the four consumers? Our proposed Consumer Surplus definition will sustain any type of allocation.

We note that for stochastic demand functions, the Consumer Surplus $CS(\epsilon)$ will be a function of the noise realization ϵ. An upper bound on the Consumer Surplus can be obtained by considering the case with infinite supply, that is, $\int_0^{d(p,\epsilon)} [d^{-1}(w, \varepsilon) - p]dw$. In reality, as we mentioned, we have to account for situations where demand exceeds supply. In particular, the actual Consumer Surplus will be a fraction of this upper bound, based on the proportion of customers who are served. The way of formalizing this precise proportion of served customers depends on the specific rationing capacity rule under consideration.

2.2.1. Rationing Capacity Rules. In practice, when suppliers receive an unexpectedly large amount of orders, they need to decide how to allocate their supply. Even if the supply allocation rule is not chosen by the seller, it is important to account for the way products are allocated when computing the Consumer Surplus. In this context, one can consider several rules for allocating available quantities. In most settings, suppliers do not have access to customer valuations and simply assign supply to the first customers that show up to their stores. Examples include car dealers, fashion, electronic products, and online shopping. In addition, customers' arrival times are often independent of valuations for the product and, hence, we label this rule with the superscript R (Random allocation). In other words, all potential customers who are interested in purchasing the product, have the same likelihood to be served. Two additional common rules are H (Highest willingness to pay) and L (Lowest willingness to pay). As their names indicate, available supply is allocated to the consumers with the highest (resp. lowest) willingness to pay,[5] and discard the consumers with the lowest (resp. highest) valuations. Note that these two allocation rules are the best (resp. worst) in terms of the total customer welfare. Besides these three allocation rules, one can consider alternative rules. More precisely, a general allocation rule $\mathcal{A}$ can be any allocation of available capacity q at price p to the consumers for a given demand realization $d(p, \epsilon)$. Mathematically, an allocation $\mathcal{A}$ is defined as a function[6] $\mathcal{A}: [0, d(p, \varepsilon)] \rightarrow [0, 1]$ so that for any p, q, and ϵ, we have:

$$\int_0^{d(p,\varepsilon)} \mathcal{A}(w)dw = \min\{d(p, \varepsilon), q\}. \tag{2}$$

Then, for any $w \in [0, d(p, \epsilon)]$, $\mathcal{A}(w)$ can be interpreted as the likelihood that an infinitesimal consumer receives the product. Equation (2) ensures that the total supplied units under the allocation rule A are equal to the volume of sales (i.e., the minimum between demand and supply). Thus, given p, q, and ϵ, the allocation rules functions $\mathcal{A}^H$, $\mathcal{A}^L$, $\mathcal{A}^R$ are $\mathcal{A}^H(w) = 1_{\{w \le q\}}$, $\mathcal{A}^L(w) = 1_{\{w \ge d(p,\varepsilon)-q\}}$, and $\mathcal{A}^R(w) = \min\left\{1, \frac{q}{d(p,\, \epsilon)}\right\}$ respectively. Consequently, given an allocation rule, the Consumer Surplus can be defined as the sum of the surplus over consumers who are willing to make a purchase, weighted by their likelihood of receiving the product. Although this definition holds for any allocation rule, our analysis will focus on the three most popular rules H, L, and R.

2.2.2. Definition and Graphical Interpretation. In this section, we present the Consumer Surplus definition under a general stochastic demand function $d(p, \epsilon)$. We first present the expression for a general allocation rule $\mathcal{A}$ and then for the three aforementioned rules (H, L, and R). In addition, we provide a graphical interpretation by illustrating the definitions for the case of a linear demand with additive noise. A similar methodology and graphical intuitions can be found in Visscher (1973) for the total welfare (the sum of consumer and supplier surpluses).

For a general allocation rule $\mathcal{A}$, the Consumer Surplus for any given ϵ and (p,q) is given by:[7]

$$CS^{\mathcal{A}}(\varepsilon) = \int_0^{d(p,\varepsilon)} [d^{-1}(w, \varepsilon) - p]\mathcal{A}(w)dw. \tag{3}$$

Under such a general allocation rule, it is often impractical to measure the Consumer Surplus. We thus consider the three special rules, H, L, and R. As discussed, the H and L rules can be seen as best- and worst-cases, respectively in terms of the Consumer Surplus value (this is formally shown in Observation 1 below).

For the H rule, the Consumer Surplus for any given ϵ and (p, q) is given by:

$$CS^H(\varepsilon) = \int_0^{d(p,\varepsilon)} [d^{-1}(w, \varepsilon) - p]1_{\{w \le q\}}dw.$$

This rule is the best possible situation for the pool of consumers, as customers with high valuations typically want the product the most and are served with the highest priority. In practice, this setting can occur when customers are passionate and eager to buy the product (e.g., concert tickets, new generation of technology products). In the case where demand exceeds supply (i.e., $d(p, \epsilon) > q$), the $d(p, \epsilon) - q$ customers with the lowest valuations are not served and, hence, the surplus of those consumers is 0 (an illustration is shown in the left panel of Figure 2).

For the L rule, the Consumer Surplus for any given ϵ and (p, q) is given by:

$$CS^L(\varepsilon) = \int_0^{d(p,\varepsilon)} [d^{-1}(w, \varepsilon) - p]1_{\{w \ge d(p,\varepsilon)-q\}}dw.$$

This rule is the worst possible situation for consumers, as customers with high valuations are served with the lowest priority. In practice, this setting can correspond to the case where low-valuation customers are deal seekers and arrive first to the store (e.g., promotional events or flash sales). In this case, when demand exceeds supply (i.e., $d(p, \epsilon) > q$),

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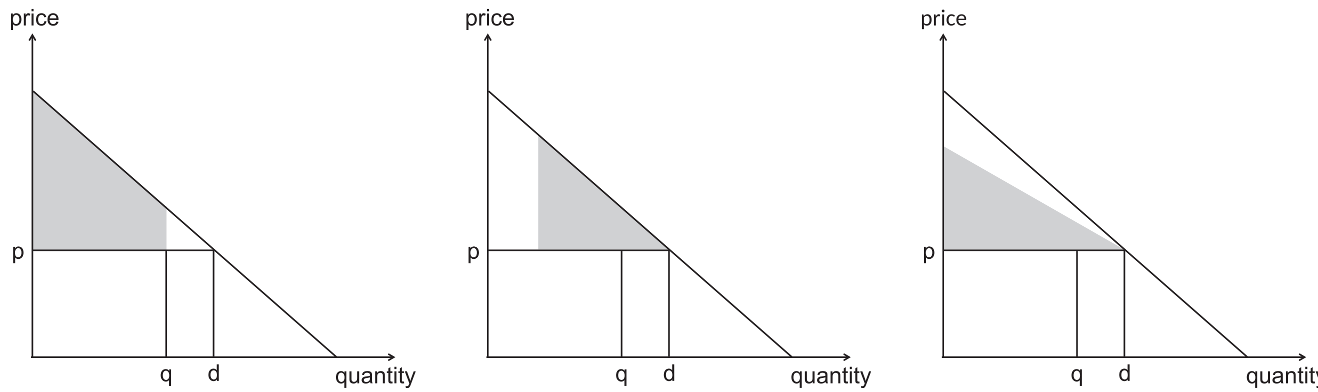
Figure 2 Left: High Willingness to Pay Allocation Rule. Center: Low Willingness to Pay. Right: Random Allocation Rule. The Shaded Region Represents the Consumer Surplus for the Case When Demand Exceeds Supply



the $d(p, \epsilon)$–q customers with the highest valuations are not served (see the center panel of Figure 2).

For the R rule, the Consumer Surplus for any given ϵ and (p, q) is given by:

$$CS^R(\varepsilon) = \int_0^{d(p,\varepsilon)} [d^{-1}(w, \varepsilon) - p] \min\left\{1, \frac{q}{d(p, \varepsilon)}\right\} dw. \quad (4)$$

Under the R rule, customers arrive at random irrespective of their willingness to pay for the product. For certain demand realizations, some proportion of these customers will not be served due to stock-outs. The proportion of served customers is given by the ratio of actual sales over potential demand, that is, $\min\{1, \frac{q}{d(p, \varepsilon)}\}$. Thus, the Consumer Surplus can be defined as the total available surplus times the proportion of served customers. In this case, the Consumer Surplus can be depicted as the grey area between the inverse demand and the price (see the right panel of Figure 2). Equivalently, when demand exceeds supply, each infinitesimal consumer has a surplus weighted by $q/d(p, \epsilon)$. We note that the case of a random allocation rule with an additive noise for a single product was already considered in the literature (see, e.g., Cohen et al. 2015, Raz and Ovchinnikov 2015).

We highlight that all the above definitions coincide with the deterministic definition in Equation (1) when the noise vanishes (i.e., $\epsilon = 0$ and $\epsilon = 1$ for additive and multiplicative noises respectively) and when $q = E[d(p, \varepsilon)]$. In section 2.3, we study the impact of demand uncertainty on consumers by comparing CS_{det} to the expected Consumer Surplus $E[CS^A(\varepsilon)]$ for the H, L, and R allocation rules. It allows us to infer what would happen if one is ignoring demand uncertainty and stock-outs when measuring customer welfare. Finally, note that for any allocation rule, the following property holds.

OBSERVATION 1. *For any A, p, q, and ϵ, the following holds:*

$$CS^L(\varepsilon) \leq CS^A(\varepsilon) \leq CS^H(\varepsilon).$$

The proof of Observation 1 can be found in the appendix. As a result, we also have $E[CS^L(\varepsilon)] \leq E[CS^A(\varepsilon)] \leq E[CS^H(\varepsilon)]$. Consequently, by studying the H and L rules, we cover the best and worst cases for consumers.

As discussed, the demand function comes from the result of a utility-maximization problem solved by a representative consumer. Consequently, the source of uncertainty in the demand function also comes from the corresponding uncertainty in the utility function. We emphasize that the underlying utility model—along with its own source of uncertainty—gives rise to the stochastic demand function we consider. Another way to derive the Consumer Surplus is by directly considering the representative consumer utility function denoted $u(v)$. Interestingly, one can derive the same expressions for the Consumer Surplus under stochastic demand from a utility perspective (the details are omitted for conciseness). In summary, we defined the Consumer Surplus for a stochastic demand function with a general capacity rule. At the same time, it extends the current understanding of consumer utility and its connection to the Consumer Surplus.

2.3. Impact of Demand Uncertainty on Consumers

Our goal is to investigate to what extent ignoring stock-outs and demand uncertainty will lead to a miscalculation of the Consumer Surplus. As discussed, this question can be of interest to policy makers when assessing the customer welfare of specific policies or public interventions. We consider both additive and

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multiplicative noises and the three allocation rules discussed in Section 2.2.1. To simplify the analysis and to isolate the effect of demand uncertainty, we first assume that the market price is the same in both the deterministic and stochastic settings. This can be motivated by the fact that the supplier is serving two different markets (e.g., selling the same product in two different countries) and must set the same selling price due to regulations or marketing considerations. For example, studying the impact of demand uncertainty can be helpful when the supplier plans to enter a new market for which sales data are very limited. We then extend our results to the more general setting where prices are exogenously determined.

2.3.1. Exogenous Pricing. When demand is deterministic, the supplier naturally matches supply with demand, namely, $q^0 = d(p^0)$, where p^0 denotes the market price. When demand is stochastic, we assume that the price is still equal to p^0 but the quantity q^{sto} can differ from $d(p^0)$. For instance, the quantity could be set according to the optimal newsvendor ordering (see, e.g., Petruzzi and Dada 1999, Porteus 1990) or according to an alternative ordering policy.

We first consider the case of a multiplicative noise, $d(p, \epsilon) = \epsilon d(p)$. The result on the impact of demand uncertainty on consumers is summarized in the following proposition.

PROPOSITION 1. *Consider a stochastic demand function with a multiplicative noise under any given allocation rule A. We then have*

$$\mathbb{E}[CS^A(\varepsilon)] \leq CS_{det}.$$

Proof. We show the result for the H rule. Then, by relying on Observation 1, we conclude that the result also holds for any allocation rule. We have

$$\begin{aligned}\mathbb{E}[CS^H(\varepsilon)] &= \mathbb{E}\left[\int_0^{d(p^0,\varepsilon)} (d^{-1}(w,\varepsilon) - p^0) 1_{\{w \leq q^0\}}\, dw\right] \\ &= \mathbb{E}\left[\int_0^{\varepsilon d(p^0)} \left(d^{-1}\left(\frac{w}{\varepsilon}\right) - p^0\right) 1_{\{w \leq q^0\}}\, dw\right] = E\left[\int_0^{d(p^0)} (d^{-1}(v) - p^0) 1_{\{v\varepsilon \leq q^0\}} \varepsilon\, dv\right] \\ &\leq \mathbb{E}\left[\int_0^{d(p^0)} (d^{-1}(v) - p^0)\varepsilon\, dv\right] = \int_0^{d(p^0)} (d^{-1}(v) - p^0)\, dv = CS_{det},\end{aligned}$$

where the last equality follows from $\mathbb{E}[\varepsilon] = 1$.

Interestingly, the result holds for any q^{sto}, any noise distribution, and any allocation rule. Therefore, no matter how sophisticated the supplier is in terms of allocating available supply, the consumers are always hurt when demand is stochastic relative to the deterministic setting (for a multiplicative noise). Indeed, positive demand shocks (i.e., $\epsilon > 1$) bring additional consumers but their valuations will not increase by much. This is especially true for customers who value the product the most. In particular, the maximal valuation, $d^{-1}(0)$, remains the same regardless of the noise realization, since the inverse demand function at $q = 0$ is not affected by the noise realization. This observation does not hold for an additive noise.

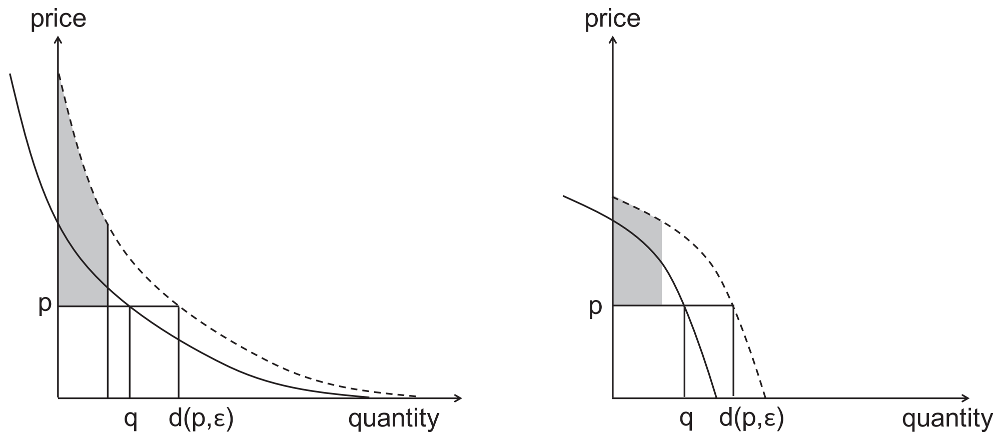
We next consider the model with an additive noise, $d(p, \epsilon) = d(p) + \epsilon$. In this case, the impact of demand uncertainty on consumers depends on three different factors: (i) the convexity properties of the nominal demand function $d(p)$, (ii) the allocation rule, and (iii) the relation between $d(p^0)$ and q^{sto}. The results are summarized in the following proposition.

PROPOSITION 2. *Consider a stochastic demand function with an additive noise and the three allocation rules, H, L, and R. We then have the following results:*

- ***H rule:*** *If $q^{sto} \geq d(p^0)$ and $d^{-1}(\cdot)$ is convex,*
$$\mathbb{E}[CS^H(\varepsilon)] \geq CS_{det}.$$
- ***L rule:*** *If $q^{sto} \leq d(p^0)$,*
$$\mathbb{E}[CS^L(\varepsilon)] \leq CS_{det}.$$
- ***R rule:*** *If $q^{sto} \leq d(p^0)$ and $d^{-1}(\cdot)$ is concave,*
$$\mathbb{E}[CS^R(\varepsilon)] \leq CS_{det}.$$

Note that in each case, the conditions are only sufficient. In other words, if one of the conditions is not satisfied, we can find examples in which the inequality can be in either direction. It follows that the impact of demand uncertainty on consumers (under an additive noise) is driven by both the allocation rule and the demand convexity or concavity. An interesting special case is linear demand with $q^{sto} = d(p^0)$ (i.e., the supplier produces according to the expected demand value). In this case, the impact of demand uncertainty on consumers crucially depends on the ability of the supplier to identify and discriminate consumers. In particular, under the H rule (best scenario for consumers), the consumers are better off when demand is stochastic, whereas this conclusion is reversed under the R or L rule. Regarding the convexity properties, a convex demand (and thus a convex inverse demand) will typically result in a higher surplus gain for consumers. This follows from the fact that positive demand shocks will induce additional customers with much higher valuations relative to the deterministic nominal demand. We illustrate this effect in the left panel of Figure 3, where the shaded area

Figure 3 **Left: Convex Inverse Demand Curve. Right: Concave Inverse Demand Curve. The Solid Line Represents the Nominal Demand Function, whereas the Dashed Line Corresponds to the Demand Function with an Additive Positive Shock. The Shaded Area Represents the Potential Additional Consumer Surplus**



represents the surplus of these additional customers assuming they get served. However, under a concave demand (and hence a concave inverse demand), positive demand shocks will introduce additional consumers with valuations only slightly higher relative to the deterministic setting, see the right panel of Figure 3.

In addition, if q^{sto} is thought in terms of the optimal newsvendor quantity, the condition $q^{sto} \leq d(p^0)$ translates into $F^{-1}(\frac{p^0-c}{p^0}) \leq 0$ (under an additive noise), where $F(\cdot)$ is the CDF of the noise ϵ and c is the (constant) marginal production cost. This condition is thus satisfied for products with low profit margins. In particular, if ϵ is symmetric, the condition reduces to $c \leq p^0 \leq 2c$.

One can also consider an alternative setting where the supplier produces q^0 units in both the deterministic and stochastic cases, where q^0 is not necessarily equal to $d(p^0)$. Such a setting may correspond to situations where suppliers need to make capacity decisions before knowing the demand realization. In this case, we can extend most of the results presented in this section (the results are omitted for conciseness). Nevertheless, it seems reasonable to assume that when demand is deterministic, we have $q^0 = d(p^0)$ as the supplier can tailor its production to exactly match demand.

2.3.2. Endogenous Pricing. We next consider the case where prices are endogenously determined. In reality, market prices can be set by solving a revenue-maximization problem. Consequently, the equilibrium prices in both settings (deterministic and stochastic) may possibly differ and, thus, the Consumer Surplus will also differ. To capture the endogenous nature of prices, we incorporate the seller's pricing and production strategy into the revenue-maximization problem. We consider a marginal production cost $c > 0$. For the deterministic setting, the seller solves the following problem:

$$\max_p (p-c)d(p). \tag{5}$$

We denote by p^d the maximizer of problem (5) and $q^d = d(p^d)$. Then, the Consumer Surplus can be simply computed by using the expression $CS_{det} = \int_0^{d(p)}(d^{-1}(w)-p)dw$. For the stochastic setting, the optimization problem becomes

$$\max_{p,\,q} p E[\min\{d(p,\,\varepsilon),\,q\}] - cq, \tag{6}$$

which is the price-setting newsvendor problem. We denote (p^s,q^s) the optimal solution of problem (6). Before stating the relation between the Consumer Surplus in both settings, we first state a well-known result on the relationship between the optimal prices from problems (5) and (6) for additive and multiplicative noises (Karlin and Carr 1962, Mills 1959, Salinger and Ampudia 2011).

LEMMA 1. *Under an additive noise $p^d \geq p^s$, whereas under a multiplicative noise $p^d \leq p^s$.*

The next proposition compares the Consumer Surplus in the deterministic and stochastic settings under a multiplicative noise when prices are endogenous.

PROPOSITION 3. *Under a stochastic demand function with a multiplicative noise and any capacity allocation rule A, we have*

$$E[CS^A(\varepsilon)] \leq CS_{det}. \tag{7}$$

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Thus, consumers are always better off in the deterministic setting not only under the same (exogenous) prices, but also when the optimal prices and quantities are endogenously determined. The next proposition summarizes the results for an additive demand noise.

PROPOSITION 4. *Consider a stochastic demand function with an additive noise and the H allocation rule such that* $q^s \geq d(p^s)$ *and* $d^{-1}(\cdot)$ *is convex. We then have*

$$\mathbb{E}[CS^H(\varepsilon)] \geq CS_{det}. \quad (8)$$

Note that in the case of an additive demand noise, unlike the case of exogenous prices (i.e., Proposition 2), consumers are not always worse off under the R or L rule when prices are endogenously determined (we formally identified counterexamples). The rationale behind this result is that uncertainty under an additive noise induces a lower price, hence offsetting the inequalities for the R and L rules outlined in Proposition 2 for the case with exogenous prices.

We highlight that there is no clear way to compare the Consumer Surplus under exogenous and endogenous pricing. Indeed, the results will highly depend on the value of the exogenous price. Consider, for example, the setting with a single product. Let p_0 denote the price in the exogenous setting. If p_0 is very high, it is clear that the Consumer Surplus will be lower under exogenous pricing, whereas if p_0 is very low, then the Consumer Surplus will be higher under exogenous pricing. In the revised paper, we have now explicitly mentioned the fact that this comparison highly depends on the value of the exogenous price.

3. Multiple Products

In this section, we consider a setting with $n \geq 2$ products. The demand function, $d(\mathbf{p}, \boldsymbol{\varepsilon}) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n_+$, is expressed as a function of the price vector $\mathbf{p} \in \mathbb{R}^n$ and a random vector $\boldsymbol{\varepsilon} \in \mathbb{R}^n$ with support $\Omega \subset \mathbb{R}^n$. We assume that $d(\mathbf{p}, \boldsymbol{\epsilon})$ is continuous in $\mathbf{p}$ and $\boldsymbol{\epsilon}$ and differentiable almost everywhere with respect to $\mathbf{p}$.[8] As before, we consider the following two cases: (i) additive noise: $d(\mathbf{p}, \boldsymbol{\epsilon}) = d(\mathbf{p}) + \boldsymbol{\epsilon}$ and (ii) multiplicative noise: $d(\mathbf{p}, \boldsymbol{\epsilon}) = \mathbf{D}_{\boldsymbol{\epsilon}} d(\mathbf{p})$, where $\mathbf{D}_{\boldsymbol{\epsilon}}$ refers to the diagonal matrix with the elements of $\boldsymbol{\epsilon}$ in its diagonal. As stated in Krishnan (2010), for a demand with multiplicative noise, the Slutsky symmetry conditions (defined formally below) can only be met if the realizations of the random variables ϵ_i are identical across all products, that is, $\mathbf{D}_{\boldsymbol{\varepsilon}} = \varepsilon_1 \mathbf{I}$ where $\mathbf{I} \in \mathbb{R}^{n \times n}$ is the identity matrix. Throughout this paper, when referring to the setting with multiple products under multiplicative noise, we impose $\mathbf{D}_{\boldsymbol{\varepsilon}} = \varepsilon_1 \mathbf{I}$. We assume that $E[\varepsilon_i] = 0$ and $E[\varepsilon_i] = 1$ for all $i \in \{1, \ldots, n\}$ in the additive and multiplicative cases, respectively. We also assume that the demand for each product is decreasing in its own price and non-decreasing in the other prices (i.e., $\frac{\partial d_i}{\partial p_i} < 0$ for all $i \in \{1, \ldots, n\}$ and $\frac{\partial d_i}{\partial p_j} \geq 0$ for all $i, j \in \{1, \ldots, n\}, i \neq j$). This is a common assumption that captures the fact that the products are substitutable goods (e.g., two competing brands in the same category). Modeling the substitutability behavior of consumers by using this type of demand models is common in the literature (see, e.g., Cohen and Perakis 2020, Cohen et al. 2020, Pindyck and Rubinfeld 2018). We highlight that the substitution behavior is captured by the cross-price effects present in the demand function (here, substitution refers to customers switching products based on price variation, as opposed to focusing on stock-out events). We note that in a model where the demand of each product is a function of all products' prices, stock outs are not explicitly captured in the demand model. However, we formally capture the stock-out events by inputting the minimum between demand and supply, namely, $\min\{d_i, q_i\}$ for each product $i =$- $1, \ldots, n$. Thus, consumers are still substituting among the products based on pricing considerations; but the demand corresponds to the total number of demanded units, as opposed to the sales. To account for potential stock outs, we truncate the demand by taking the minimum between demanded units and available supply. In the appendix, we consider and analyze an alternative model that explicitly captures substitution into the consumer utility-maximization problem by accounting for inventory constraints. This modeling framework naturally leads to a model in which the outcome consumption results in a demand function that allows for substitution among products in the events of stock-outs. Interestingly, this alternative utility model leads to the same results and insights presented in this section. In addition, we assume that the price change of a particular product has a stronger effect on its own demand realtive to the sum of the price changes of the other products (i.e., $\sum_{j \neq i} \left| \frac{\partial d_i}{\partial p_j} \right| < \left| \frac{\partial d_i}{\partial p_i} \right|$ for all $j \in \{1, \ldots, n\}$). This assumption is called the *strict diagonal dominance* condition and is common in the literature (see, e.g., Arrow and Hahn 1971). The above assumptions imply that the negative of the demand Jacobian (with respect to prices) is a non-singular M-matrix and, hence, the Jacobian of the inverse demand is non-positive with strictly negative elements in its diagonal.

3.1. Deterministic Demand and Slutsky Conditions

In the deterministic setting, the demand vector is a function only of the price vector,[9] that is,

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$d(\mathbf{p}):\mathbb{R}^n \to \mathbb{R}^n_+$. In this case, we naturally have $d(\mathbf{p}) = \mathbf{q}$. Note that for a given $\mathbf{q}$, the inverse demand function $d^{-1}(\mathbf{q})$ represents the maximal price vector for which consumers will demand $\mathbf{q}$ units (i.e., consumers' willingness to pay). This interpretation is important as we will compute the Consumer Surplus by integrating consumers' willingness to pay over the quantity space. Alternatively, one can integrate over the price space, as in [1000] where the author computes the Consumer Surplus for multiple products under deterministic demand. In this case, the Consumer Surplus is expressed as the path integral over the sum of the demand functions of each product. Thus, the value of the integral may be path dependent if the Slutsky conditions are not satisfied.[10] To avoid the undesired path-dependence property, we assume that the Slutsky conditions are satisfied. We acknowledge that this condition is somewhat restrictive. However, if we were to relax the Slutsky condition, then it would not be possible to adequately measure the Consumer Surplus (for a setting where the demand function comes from a representative consumer utility-maximization problem). To our knowledge, the Slutsky condition has been imposed in all the studies that consider a formal demand function that comes from a utility maximization problem faced by a representative consumer. As otherwise, measuring the Consumer Surplus (even in a deterministic setting) is impossible. In this paper, we will compute the Consumer Surplus as the path integral over the inverse demand for cases where there may be a mismatch between demand and supply. As mentioned, when demand is stochastic, the produced units can sometimes be lower than demand. As in the single-product setting, we address this issue by introducing an n-dimensional allocation rule that assigns the available supply to customers (see section 3.2 for more details). We next write the Consumer Surplus as the path integral over the inverse demand function:

$$CS_{det} = \int_{\mathcal{C}} [d^{-1}(r) - \mathbf{p}] \cdot dr, \tag{9}$$

where $\mathcal{C}$ is an integration path from $\mathbf{0} \in \mathbb{R}^n$ to $\mathbf{q} = d(\mathbf{p})$ and defined by the parametric function $r:[a, b] \to \prod_{i=1}^n [0, q_i]$ which is continuous and differentiable almost everywhere. The expression in Equation (9) is uniquely determined (i.e., path independent), if the cross derivatives of the inverse demand function (or demand function) are equal, see Tirole (1988). Otherwise, the expression in Equation (9) would depend on the path of integration. Note that in the single-product setting, such an issue does not exist, as the path moves along a unique direction on a segment.

So far, we assumed that demand and supply are matching (i.e., $d(\mathbf{p}) = \mathbf{q}$). Nevertheless, in several applications, this may not be the case. This motivates us to study the more general setting when supply and demand do not necessarily match (i.e., $d(\mathbf{p}) \neq \mathbf{q}$).

3.2. Stochastic Demand

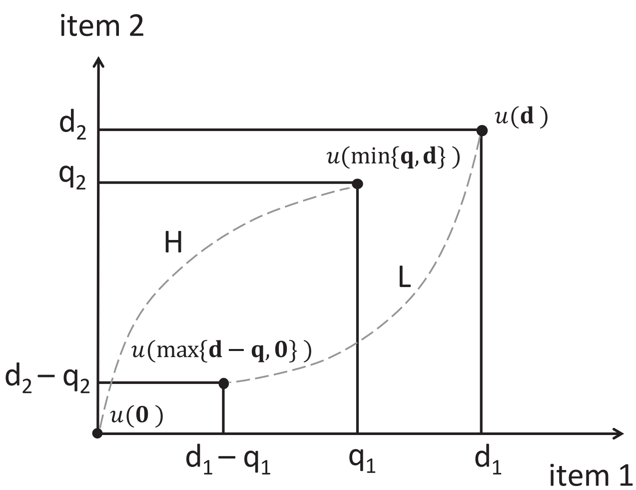
Consider vectors $\mathbf{p}$ and $\mathbf{q}$. We do not necessarily impose $\mathbf{q} = E[d(\mathbf{p}, \boldsymbol{\varepsilon})]$. As discussed, when demand is stochastic, there may be cases where production quantities are lower than the demand for each product. Our goal is to define the Consumer Surplus for multiple products, while accounting for potential stock-outs. Since the multiple-product setting is more intricate than the single-product setting, for ease of exposition, we first present the definition for the H and L rules from a utility perspective. We will then write these expressions as a function of the inverse demand. As discussed, the H and L rules correspond to the best- and worst-case scenarios in terms of Consumer Surplus and, hence, can serve as identifying performance bounds. Finally, we will consider the R rule and generalize to any allocation rule.

Under the H allocation rule, if the available units $\mathbf{q}$ are lower than the demand $d(\mathbf{p},\boldsymbol{\epsilon})$, a portion of the utility will not be captured by consumers. Following a similar argument as in the single-product case, the utility under this allocation will be captured by the first consumers on each item and, thus, the Consumer Surplus is $u(\min\{\mathbf{q}, d(\mathbf{p}, \boldsymbol{\varepsilon})\}) - u(\mathbf{0}) - \mathbf{p}^{\mathbf{T}}\min\{\mathbf{q}, d(\mathbf{p}, \boldsymbol{\varepsilon})\}$, where $u(\mathbf{0}) = 0$. The H rule implies that among the $d_i(\mathbf{p}, \boldsymbol{\varepsilon})$ consumers who are willing to purchase product i, only the consumers with the highest valuations will be served (i.e., the first $\min\{d_i(\mathbf{p}, \boldsymbol{\varepsilon}), q_i\}$ customers). For the L rule, as in the single-product case, the utility of consumers is captured by the last units consumed, namely, the consumers from $\max\{d(\mathbf{p},\boldsymbol{\epsilon})-\mathbf{q},\mathbf{0}\}$ to $d(\mathbf{p},\boldsymbol{\epsilon})$ and, thus, the Consumer Surplus under the L rule is $u(d(\mathbf{p}, \boldsymbol{\varepsilon})) - u(d(\mathbf{p}, \boldsymbol{\varepsilon}) - \max\{\mathbf{q}, \mathbf{0}\}) - \mathbf{p}^{\mathbf{T}}\min\{\mathbf{q}, d(\mathbf{p}, \boldsymbol{\varepsilon})\}$. We illustrate these two definitions for a setting with two products in Figure 4.

We note that the utility-maximization problem leads to $d^{-1} = \nabla_{\mathbf{v}} u$, so that the Consumer Surplus can be written as a function of the inverse demand. For the H rule, the integral of the inverse demand should go from $\mathbf{0}$ to $\min\{\mathbf{q}, d(\mathbf{p}, \boldsymbol{\epsilon})\}$. The latter can be expressed using an integration path from $\mathbf{0}$ to $d(\mathbf{p},\boldsymbol{\epsilon})$, while weighting the inverse demand with the corresponding allocation. We denote by C^{ε} the path from $\mathbf{0}$ to $d(\mathbf{p},\boldsymbol{\epsilon})$ and by $r^{\boldsymbol{\varepsilon}}$ the corresponding parametric function along the path.[11] Then, the integral path for the H rule goes from $\mathbf{0}$ to $\min\{\mathbf{q},d(\mathbf{p},\boldsymbol{\epsilon})\}$ with a weight of 1, whereas the remaining path goes from $\min\{\mathbf{q},d(\mathbf{p},\boldsymbol{\epsilon})\}$ to $d(\mathbf{p},\boldsymbol{\epsilon})$ with a weight of 0. We then have

$$CS^H(\boldsymbol{\varepsilon}) = \int_{C^{\varepsilon}} \sum_{i=1}^n [d_i^{-1}(r^{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon}) - p_i] 1_{\{r_i^{\boldsymbol{\varepsilon}} \leq q_i\}} dr_i^{\boldsymbol{\varepsilon}}, \tag{10}$$

Figure 4 Example with $n = 2$ on How to Compute the Consumer Surplus for the H and L Rules



where C^ϵ is any path consistent with the H rule in the sense that it crosses the point $\min\{\mathbf{q},d(\mathbf{p},\boldsymbol{\epsilon})\}$. Note that the expression in Equation (10) is path independent across all such paths. In addition, the Consumer Surplus can alternatively be written as the difference between the utility evaluated at $\min\{d(\mathbf{p}, \boldsymbol{\epsilon}), \mathbf{q}\}$ and $\mathbf{0}$ (which is consistent with the traditional deterministic interpretation).

For the L rule, the integral of the inverse demand should go from $\max\{d(\mathbf{p}, \boldsymbol{\epsilon})-\mathbf{q}, \mathbf{0}\}$ to $d(\mathbf{p},\boldsymbol{\epsilon})$. Recall that under the L rule, items are allocated to customers with the lowest valuations (among all customers with a valuation above price). Namely, among the $d_i(\mathbf{p}, \boldsymbol{\epsilon})$ customers who demand product i, only the $\min\{d_i(\mathbf{p}, \boldsymbol{\epsilon}), q_i\}$ customers with the lowest valuations will be served. Equivalently, we can consider any path C^ϵ from $\mathbf{0}$ to $\max\{d(\mathbf{p}, \boldsymbol{\epsilon})-\mathbf{q}, \mathbf{0}\}$ with a weight of 0 and, then, from $\max\{d(\mathbf{p}, \boldsymbol{\epsilon})-\mathbf{q}, \mathbf{0}\}$ to $d(\mathbf{p},\boldsymbol{\epsilon})$ with a weight of 1. As a result, we can write

$$CS^L(\boldsymbol{\epsilon}) = \int_{C^\epsilon} \sum_{i=1}^{n} [d_i^{-1}(r^{\boldsymbol{\epsilon}}, \boldsymbol{\epsilon}) - p_i] 1_{\{r_i^{\boldsymbol{\epsilon}} \geq d_i(\mathbf{p},\boldsymbol{\epsilon}) - q_i\}} dr_i^{\boldsymbol{\epsilon}}. \quad (11)$$

We note that the Consumer Surplus expressions for the H and L rules, in Equations (10) and (11), have an indicator term inside the integral which assigns a weight to the marginal utility. This term varies depending on the allocation. Under the R rule, all (infinitesimal) customers have the same likelihood of receiving the item. This translates into having an allocation term of $\min\{1, \frac{q_i}{d_i(\mathbf{p}, \boldsymbol{\epsilon})}\}$ for each product i. Then, the Consumer Surplus for the R rule can be expressed as

$$CS^R(\boldsymbol{\epsilon}) = \int_{C^\epsilon} \sum_{i=1}^{n} [d_i^{-1}(r^{\boldsymbol{\epsilon}}, \boldsymbol{\epsilon}) - p_i] \min\left\{1, \frac{q_i}{d_i(\mathbf{p}, \boldsymbol{\epsilon})}\right\} dr_i^{\boldsymbol{\epsilon}}, \quad (12)$$

where C^ϵ is the path that follows a straight line from $\mathbf{0}$ to $d(\mathbf{p},\boldsymbol{\epsilon})$. Intuitively, the allocation term inside the integral in Equation (12) assigns a weight to the marginal utility according to the ratio of available quantities and demand. This allocation can be seen as the limiting case of the H (or L) rule. A more detailed explanation of this limit interpretation is provided in the appendix.

As in section 2, the treatment can be extended to a general allocation rule A. To this end, we need to specify: (i) an allocation function $\mathcal{A}: \prod_{j=1}^{n} [0, d_j(\mathbf{p}, \boldsymbol{\epsilon})] \to [0, 1]^n$ and (ii) an integration path C^ϵ represented by a parametric function $r^{\boldsymbol{\epsilon}}$ from $\mathbf{0}$ to $d(\mathbf{p},\boldsymbol{\epsilon})$ that satisfies for each i:

$$\int_{C^\epsilon} \mathcal{A}_i(r^{\boldsymbol{\epsilon}}) dr_i^{\boldsymbol{\epsilon}} = \min\{q_i, d_i(\mathbf{p}, \boldsymbol{\epsilon})\}. \quad (13)$$

Equation (13) implies that the total number of allocated units for each product is equal to the minimum between demand and supply. As a result, we obtain:[12]

$$CS^A(\boldsymbol{\epsilon}) = \int_{C^\epsilon} \sum_{i=1}^{n} [d_i^{-1}(r^{\boldsymbol{\epsilon}}, \boldsymbol{\epsilon}) - p_i] \mathcal{A}_i(r^{\boldsymbol{\epsilon}}) dr_i^{\boldsymbol{\epsilon}}. \quad (14)$$

Finally, the expected Consumer Surplus can be obtained by taking the expectation over $\boldsymbol{\epsilon}$:

$$E[CS^A(\boldsymbol{\epsilon})] = \int_\Omega CS^A(\boldsymbol{\epsilon}) dF(\boldsymbol{\epsilon}).$$

As in the single-product setting, we can also derive the above Consumer Surplus definitions from a utility perspective (the details are omitted). As before, the Consumer Surplus under the different allocation rules satisfies the following ordering.

Observation 2. *For any A, $\mathbf{p}$, $\mathbf{q}$, $\boldsymbol{\epsilon}$, and C^ϵ defined by $r^{\boldsymbol{\epsilon}}$, the following holds:*

$$CS^L(\boldsymbol{\epsilon}) \leq CS^A(\boldsymbol{\epsilon}) \leq CS^H(\boldsymbol{\epsilon}).$$

The proof of Observation 2 can be found in the appendix. Consequently, we have $E[CS^L(\boldsymbol{\epsilon})] \leq E[CS^A(\boldsymbol{\epsilon})] \leq E[CS^H(\boldsymbol{\epsilon})]$.

As we can see, the Consumer Surplus for multiple products under stochastic demand depends on the noise realization and on the allocation rule. When supply exceeds demand, there is no stock-out and we are back to the deterministic case. However, when demand exceeds supply, some consumers may not be served. In this case, the allocation rule will determine which customers are served, hence allowing us to properly compute the Consumer Surplus. As mentioned, the allocation rule is related to the order in

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which customers are served for a particular product. For example, the H rule translates to first serving the customers with the highest valuations.

3.2.1 Example. We next present a concrete example with two products and compute the Consumer Surplus under the three different allocation rules. We consider a linear demand function with an additive noise of the form $d(\mathbf{p},\epsilon)=\overline{\mathbf{d}}-\mathbf{B}\mathbf{p}+\epsilon$ that satisfies the conditions mentioned at the beginning of section 3,[13] and we denote by $\mathbf{p^0}$ and $\mathbf{q^0}$ the given price and quantity vectors. Table 1 reports the Consumer Surplus values under a specific noise realization. As we can see, the value of the Consumer Surplus highly depends on the allocation rule.

Having defined the Consumer Surplus for a general allocation rule under stochastic demand for multiple products, we next compare the expected Consumer Surplus to the deterministic setting. We show that the impact of demand uncertainty on consumers depends on several factors such as the convexity properties of the demand, the noise structure, and the allocation rule.

3.3. Impact of Demand Uncertainty on Consumers

In this section, we extend the analysis and results for multiple products. If the different products are independent (i.e., no cross-item effects), one can extend the results from the single-product case by simply computing the Consumer Surplus for each product separately and summing up the n terms. Nevertheless, the most interesting case is when the demand of each product depends on both its own price and the prices of other products (i.e., the price of product i also affects the demand of products $j \neq i$).

3.3.1. Exogenous Pricing. We consider a setting with n products and set the (exogenous) price vector to $\mathbf{p}^0$. As before, when demand is deterministic, the production quantities are set to match the nominal demand, namely, $\mathbf{q^0}=d(\mathbf{p^0})$. Under stochastic demand, the supplier produces quantities $\mathbf{q^{sto}}$ which do not necessarily match $d(\mathbf{p^0})$. As in the single-product case, we consider both multiplicative and additive noises and start with the setting where the prices are the same.

Table 1 Consumer Surplus Values for the Different Allocation Rules. Parameters: $\bar{d}_1=10$, $\bar{d}_2=7$, $B_{11}=B_{22}=1$, $B_{12}=B_{21}=-0.25$, $\epsilon_1=1$, $\epsilon_2=0$, $p_1=4$, $p_2=2$, $d_1=7.5$, $d_2=6$, $q_1=6.5$, and $q_2=6$

Rule	CS
High	60.667
Low	52.133
Random	56.400

Proposition 5. *Consider a stochastic demand function with a multiplicative noise and* $\mathbf{q^{sto}}=d(\mathbf{p^0})$. *Then, under any allocation rule A, we have*

$$E[CS^A(\boldsymbol{\epsilon})] \leq CS_{det}.$$

We next consider a demand model with an additive noise.

Proposition 6. *Consider n products and a stochastic demand function with an additive noise. We then have the following results:*

- ***L rule:*** *if* $\mathbf{q^{sto}} \leq d(\mathbf{p^0})$
$$E[CS^L(\boldsymbol{\epsilon})] \leq CS_{det}.$$
- ***R rule:*** *If* $d_i^{-1}(\cdot)$ *is concave for all* $i \in \{1, \ldots, n\}$, *and if* $\mathbf{q^{sto}} \leq d(\mathbf{p^0})$
$$E[CS^R(\boldsymbol{\epsilon})] \leq CS_{det}.$$
- ***H rule:*** *Consider a linear demand (i.e.,* $\mathbf{d}=\overline{\mathbf{d}}-\mathbf{B}\mathbf{p}+\epsilon$*) with independent and symmetric noises. If* $\mathbf{q^{sto}} \geq d(\mathbf{p^0})$ *and* $3\mathbf{D}_{\mathbf{B}^{-1}}-\mathbf{B}^{-1}$ *is positive semi-definite, where* $\mathbf{D}_{\mathbf{B}}^{-1}$ *is the diagonal matrix with the diagonal elements of* $\mathbf{B}^{-1}$*, then*
$$E[CS^H(\boldsymbol{\epsilon})] \geq CS_{det}.$$

We note that for the H rule, the Consumer Surplus inequality obtained in the single-product setting holds only under more restricted conditions (linear demand with independent and symmetric noises plus an additional technical condition). The technical condition outlined for the H rule in Proposition 6 is always satisfied for $n \leq 3$. To ensure that demand values remain non-negative, we consider formally imposing a non-negativity demand constraint for the case of an additive noise under the L or R rule. We then show that the results still hold after imposing such a non-negativity constraint (see more details in the proof of Proposition 6 in the appendix). In summary, we have shown that in many cases, demand uncertainty hurts consumers. Our results also suggest that ignoring uncertainty may severely impact the Consumer Surplus value. In section 3.4, we will show that for a linear demand with an additive noise, under the R rule, the expected Consumer Surplus can be as far as 50% from the deterministic setting.

3.3.2. Endogenous Pricing. As in the single-product setting, we next consider the case with endogenous prices. In particular, we look at the case where n firms compete in a price-newsvendor setting. Note that the deterministic demand case can be seen as a special case when the quantities to produce exactly match the demand at the equilibrium price. Each firm i has a marginal cost c_i and solves the following problem:

$$\max_{p_i, q_i} p_i \mathbb{E}\big[\min\{d_i(\mathbf{p}, \boldsymbol{\varepsilon}), q_i\}\big] - c_i q_i.$$

To the best of our knowledge, the counterpart of Lemma 1 for multiple products has not been previously established. It seems that such a general result cannot be derived due to the lack of tractability. To gain analytical tractability, we consider the case of a linear demand model with either an additive noise (i.e., $d(\mathbf{p},\boldsymbol{\epsilon}) = \bar{\mathbf{d}} - \mathbf{B}\mathbf{p} + \boldsymbol{\epsilon}$ with $E[\boldsymbol{\varepsilon}] = 0$) or a multiplicative noise (i.e., $d(\mathbf{p},\epsilon) = \epsilon(\bar{\mathbf{d}} - \mathbf{B}\mathbf{p})$ with $E[\varepsilon] = 1$) and derive this result in Lemma 2 below. The outcome of this model with an additive noise has been studied in Chen et al. (2004) and Zhao and Atkins (2008). [14] We denote $(\mathbf{p}^s, \mathbf{q}^s)$ the equilibrium price and quantity vectors. The first-order condition over the quantities leads to $q_i^s = \bar{d}_i - \sum_{j=1}^{n} B_{ij} p_j^s + F_i^{-1}(\frac{p_i^s - c_i}{p_i^s})$, where $F_i(\cdot)$ is the cdf of ϵ_i. Then, the first-order condition over the prices leads to the following fixed-point equation: $\Psi(\mathbf{p}^s) + \bar{\mathbf{d}} - \mathbf{B}\mathbf{p} - \mathbf{D}(\mathbf{p}^s - \mathbf{c}) = \mathbf{0}$, where $\Psi_i(\mathbf{p}^s) = E[\min\{\varepsilon_i, F_i^{-1}(\frac{p_i^s - c_i}{p_i^s})\}]$ and $\mathbf{D} \in \mathbb{R}^{n \times n}$ is a diagonal matrix with the diagonal elements of $\mathbf{B}$ in its diagonal and zero elsewhere. Equivalently, $\Psi(\mathbf{p}^s)$ can be written as $\Psi(\mathbf{p}^s) = E[\min\{d(\mathbf{p}^s, \boldsymbol{\varepsilon}), \mathbf{q}^s\}] - d(\mathbf{p}^s)$, thus corresponding to the expected difference between sales and demand. In the special case of a deterministic demand, the equilibrium prices can be obtained in closed form as $\mathbf{p}^\mathbf{d} = \mathbf{c} + (\mathbf{B} + \mathbf{D})^{-1}(\bar{\mathbf{d}} - \mathbf{B}\mathbf{c})$.

LEMMA 2. *Consider a linear demand function. Under an additive noise* $\mathbf{p}^\mathbf{d} \geq \mathbf{p}^\mathbf{s}$*, whereas under a multiplicative noise* $\mathbf{p}^\mathbf{d} \leq \mathbf{p}^\mathbf{s}$.

Thus, Lemma 2 extends the result of Lemma 1 for multiple products under linear demand. The next two propositions state the results for multiplicative and additive demand cases.

PROPOSITION 7. *For a linear demand function with an multiplicative noise, we have*

$$\mathbb{E}[CS^H(\varepsilon)] \leq CS_{det}.$$

PROPOSITION 8. *For a linear demand function with an additive noise, if* $3\mathbf{D}_{\mathbf{B}^{-1}} - \mathbf{B}^{-1}$ *is positive semi-definite, we have*

$$E[CS^H(\boldsymbol{\varepsilon})] \geq CS_{det}.$$

As in the single-product setting, the inequalities for the R and L rules presented for the case of endogenous prices (see Proposition 6) where consumers are better off in the deterministic setting, do not hold anymore under endogenous prices. Indeed, when prices are endogenously determined, the firms will charge a higher price in the deterministic case, as stated in Lemma 2. It can thus offset the Consumer Surplus inequality. We conclude this section by presenting a plot of the Consumer Surplus ratio (stochastic divided by deterministic) as a function of the demand uncertainty magnitude for the different settings (see Figure 5). Ultimately, our model and results allow us to quantify the impact of the demand uncertainty magnitude on the Consumer Surplus. For example, depending on the extent of the Consumer Surplus loss, firms can invest efforts in collecting additional data and in developing more sophisticated demand prediction methods to ultimately reduce demand uncertainty.

3.4. Analytical Bounds for Linear Demand

To draw additional insights on the comparison of the Consumer Surplus under stochastic and deterministic demand, we consider the special case of linear demand with an additive noise. For simplicity, we assume that all n products are symmetric and that $\mathbf{q} = d(\mathbf{p})$, that is, production quantities exactly match the nominal demand, and we consider the random allocation rule. More precisely, the demand function for the n products is given by:

$$d(\mathbf{p}, \boldsymbol{\varepsilon}) = \bar{\mathbf{d}} - \mathbf{B}\mathbf{p} + \boldsymbol{\varepsilon}. \tag{15}$$

In this case, we arrive at the following expressions:

$$\begin{aligned} CS_{det} &= [d(\mathbf{p})]^T \frac{\mathbf{B}^{-1}}{2} d(\mathbf{p}), \\ CS^R(\boldsymbol{\varepsilon}) &= [\min\{\mathbf{q}, d(\mathbf{p}, \boldsymbol{\varepsilon})\}]^T \frac{\mathbf{B}^{-1}}{2} d(\mathbf{p}, \boldsymbol{\varepsilon}), \end{aligned}$$

where the first (resp. second) equation corresponds to the Consumer Surplus for deterministic (resp. stochastic) demand. Recall that under stochastic demand, the Consumer Surplus becomes a random variable, so we are interested in $E[CS^R(\boldsymbol{\varepsilon})]$.

Since we assume that all the products are symmetric, the parameters $\bar{d}_i$, p_i, and q_i are identical for $i = 1, \ldots, n$. In addition, the matrix $\mathbf{B}$ is such that its diagonal elements are equal to b and its off-diagonal elements (cross-price sensitivity) are equal to $-\delta < 0$ (while satisfying the diagonal dominance condition $b > (n-1)\delta$). Finally, we assume that the additive

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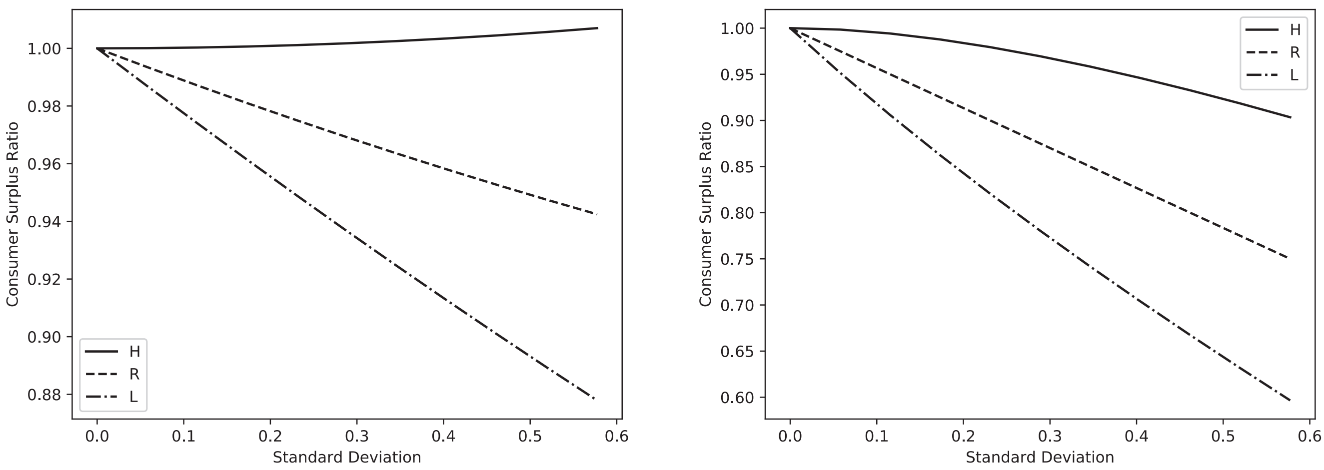
Figure 5 Consumer Surplus ratio (stochastic divided by deterministic) as a Function of the Demand Uncertainty Magnitude for Linear Demand and $n = 2$ Items. Parameters: $\bar{d}_1 = \bar{d}_2 = 5$, $B_{11} = B_{22} = 1$, $B_{12} = B_{21} = -0.4$, $c_1 = c_2 = 1$, $p_1 = p_2 = 3$, and $q_1 = q_1 = 3.2$



(a) Additive noise ($\epsilon_1, \epsilon_2 \sim U[-A, A]$)

(b) Multiplicative noise ($\epsilon \sim U[1-A, 1+A]$)

noises, $\epsilon_i \; \forall \; i = 1, \ldots, n$, are i.i.d with bounded support. In particular, we require that $\epsilon_i > -q_i$ with probability 1. For example, ϵ_i can be uniformly distributed in $[-q_i, q_i]$.

To gain analytical tractability, we consider an additive i.i.d noise with a two-point distribution (i.e., $\varepsilon_i \in \{-A, A\}$ each with probability 0.5, for some $A \le q_i$). In this case, we have

$$\begin{aligned} CS_{det} &= \frac{1}{2} n \tilde{B} q^2, \\ E[CS^R(\boldsymbol{\varepsilon})] &= \frac{1}{2} n \Big[B\, q^2 - 0.5A(B\, q - B_{ii}^{-1} A) \Big]. \end{aligned}$$

Here, $\tilde{B}$ denotes the row sum of the matrix $\mathbf{B^{-1}}$ (under symmetric products, all rows have the same sum). Also, B_{ii}^{-1} and B_{ij}^{-1} are the same for all i and $j \neq i$. We next derive a bound on the ratio of the expected Consumer Surplus under stochastic demand relative to deterministic demand.

Proposition 9. *Assume that demand is linear with an additive noise distributed according to a two-point distribution. Then, the following holds:*

1. *When the cross-price sensitivity $\delta = 0$ (i.e., independent products), we have*

$$\frac{E[CS^R(\boldsymbol{\varepsilon})]}{CS_{det}} \ge \frac{7}{8}.$$

2. *For any $0 < \delta < \frac{b}{n-1}$ (i.e., substitutable products), we have*

$$\frac{E[CS^R(\boldsymbol{\varepsilon})]}{CS_{det}} \ge \frac{1}{2} + \frac{1}{2n},$$

To illustrate Proposition 9, we consider the example with $n = 2$, $\bar{d} = 12$, $b = 1$, $\delta = 0$, $p = 2$, and $q = 12$, with noise realizations $\epsilon_1 = -2$ and $\epsilon_2 = 2$. In this example, we have $CS_{det} = 2q^2/2 = 100$ and $E[CS^R(\boldsymbol{\varepsilon})] = 0.5\min\{10, 8\} \times 8 + 0.5\min \; \{10, 12\} \times 12 = 92$, that is, a decrease of 8% relative to CS_{det}. More generally, Proposition 9 shows that when demand is linear with an additive i.i.d noise (distributed according to a two-point distribution), the expected Consumer Surplus from the random allocation can be as far as 50% from the deterministic setting.

4. Random Supply

We next examine the Consumer Surplus under stochastic supply. We directly study the multiple-product setting (the single-product setting can be obtained as a special case).

4.1. Random Supply and Deterministic Demand

We consider a deterministic demand function $d(\mathbf{p})$ with a fixed vector $\mathbf{p}$. The supply is random and represented by the function $s(\mathbf{p},\boldsymbol{\delta})$ such that $E[s(\mathbf{p}, \boldsymbol{\delta})] = d(\mathbf{p})$. As in the case of random demand, an additive supply uncertainty can be expressed as $s(\mathbf{p},\boldsymbol{\delta}) = d(\mathbf{p}) + \boldsymbol{\delta}$ with $E[\boldsymbol{\delta}] = 0$, whereas a multiplicative supply uncertainty follows $s_i(\mathbf{p}, \boldsymbol{\delta}) = \delta_i d_i(\mathbf{p})$ with $E[\delta_i] = 1$

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and $\delta_i > 0$. Unlike the stochastic demand setting, stochastic supply allows for multiple random variables under a multiplicative uncertainty.

PROPOSITION 10. *Under a stochastic supply function s* $(\boldsymbol{p}, \boldsymbol{\delta})$ *so that* $E[s(\mathbf{p}, \boldsymbol{\delta})] = d(\mathbf{p})$, *we have*

$$E[CS^H(\boldsymbol{\delta})] \leq CS_{det}.$$

Interestingly, we show that regardless of the type of noise (additive or multiplicative), the expected Consumer Surplus under stochastic supply is always lower relative to its deterministic counterpart. We next consider the general setting where both demand and supply are stochastic.

4.2. Random Supply and Random Demand

When both demand and supply are stochastic, the results will depend on the noise structure.

PROPOSITION 11. *Under multiplicative demand and supply uncertainties, we have*

$$E_{\varepsilon,\boldsymbol{\delta}}[CS^H(\boldsymbol{\varepsilon}, \boldsymbol{\delta})] \leq CS^{det}.$$

Here, the expectation is taken with respect to both noises' distributions. Interestingly, the result of Proposition 11 allows for correlated noises. As expected, the uncertainty will reduce the value of the expected Consumer Surplus for any allocation rule, when the noise is multiplicative.

PROPOSITION 12. *Under additive demand and supply uncertainties, if* d_i^{-1} *is concave, we have*

$$E_{\boldsymbol{\varepsilon},\boldsymbol{\delta}}[CS^L(\boldsymbol{\varepsilon}, \boldsymbol{\delta})] \leq CS^{det},$$
$$E_{\boldsymbol{\varepsilon},\boldsymbol{\delta}}[CS^R(\boldsymbol{\varepsilon}, \boldsymbol{\delta})] \leq CS^{det}.$$

Proposition 12 shows that the expected Consumer Surplus is lower under a stochastic setting for both the L and R rules, if d_i^{-1} is concave and the noises are additive. For the H rule, we identified examples where the inequality can go either way. Overall, this section shows that having an uncertain supply will most often hurt consumers in terms of Consumer Surplus.

5. Conclusions

A well-known concept to measure customer welfare is the Consumer Surplus. This tool was primarily developed assuming that the demand is deterministic and that products are always available. In most real-world settings, however, demand is modeled as a stochastic function of the price. While many traditional OM studies have focused on firms, several recent lines of research also account for consumers. This is especially true in sustainable operations, where customer welfare is of primary importance (e.g., Avci et al. 2014, Chemama et al. 2019, Sunar and Plambeck 2016). In this context, policy makers are often interested in assessing the impact of policies on consumers. The relevant question is then whether the standard (presumably deterministic) approach to calculate the Consumer Surplus may end up miscalculating customer welfare.

Demand uncertainty or errors in demand prediction may lead to stock-outs. In such a case, consumers who want to purchase the product may not be served, hence ultimately affecting customer welfare. We propose an extension of the Consumer Surplus which accounts for demand uncertainty and stock-outs. We first introduce a mathematical definition of an allocation rule and then present a definition of the Consumer Surplus under stochastic demand for multiple products and any allocation rule. We next use this definition to study the impact of demand and supply uncertainty on consumers. We show that demand uncertainty often hurts consumers. For example, under a demand with a multiplicative noise, consumers are always better off in the deterministic setting. Interestingly, this result holds for any demand function, noise distribution, and allocation rule. Under an additive demand noise, we show that the impact of uncertainty crucially depends on the allocation rule and on the convexity properties of the demand.

In several settings, the most practical allocation rule is the random rule, that is, consumers arrive randomly and are served independent of their valuations. For this rule, we show that consumers are typically worse off under stochastic demand. We show that when demand is linear with an additive i.i.d noise, the expected Consumer Surplus under the random allocation can be as far as 50% relative to the deterministic setting. One possible way to mitigate this Consumer Surplus loss is by using a sharing mechanism. For example, a supplier can share its excess capacity (or demand) for products among its different stores. Alternatively, competing firms may engage in a sharing inventory agreement, which can ultimately benefit consumers by increasing the Consumer Surplus under stochastic demand.

In this paper, we assumed that the substitution among products is captured by the cross-price terms in the demand function. In the appendix, we consider and analyze an alternative model that explicitly incorporates substitution into the consumer utility-maximization problem by accounting for inventory constraints. This modeling framework naturally leads to a model in which the outcome consumption results in a demand function that allows for substitution among products when stock-outs occur. Interestingly, this alternative utility model leads to the same results and insights regarding the impact of demand uncertainty on Consumer Surplus, hence supporting that our results are robust to the specific way of capturing substitution among products.

This paper is far from providing the last word on the topic of Consumer Surplus in stochastic settings and opens several opportunities for future research. For example, an interesting extension is to consider a model where the stochastic demand term depends on the price. Using the methodology developed in this work, one can compute the Consumer Surplus for settings with stochastic demand. Since customer welfare plays a growing role in many applications, one can use the tools presented in this paper to study and quantify the potential impact of government policies on consumers.

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Notes

[1] In this paper, we consider a setting where prices and quantities are set prior to the demand realization, as in the newsvendor problem.

[2] With a slight abuse of notation, we denote by $d(p,\epsilon)$ the stochastic demand function and by $d(p)$ its deterministic part. Namely, if the only argument is the price, we refer to the latter, whereas if the argument is composed of a price and a random variable ϵ, we refer to the former.

[3] The Marshallian demand function explicitly depends on price and income: it corresponds to the solution of the utility-maximization problem solved by a representative consumer who maximizes utility subject to a budget constraint. The Hicksean demand depends on price and utility: it is obtained as the solution of the expenditure-minimization problem solved by a representative consumer subject to a constraint on the minimum utility level.

[4] We assume that the nominal demand function is strictly decreasing so that its inverse exists. However, this assumption can be relaxed without altering our results. Indeed, if there is a countable disjoint set of intervals where the demand has a zero slope, then the integral in Equation (1) will have zero measure over those points.

[5] In our context, an allocation function refers to a way of mathematically expressing how available units are distributed to consumers. This is the reason why we use demand (quantity) as the argument of the allocation function. Given the monotonicity of the inverse demand, we can equivalently characterize an allocation by using the valuation (price) space, allowing us to map allocation functions to the willingness to pay of consumers.

[6] More precisely, A is a family of allocation functions parametrized by (p, q, ϵ). We omit these arguments to lighten our notation. However, our analysis carefully accounts for this dependence.

[7] Equivalently, the integral in Equation (3) can be expressed by integrating over the price space, that is, $CS^A(\varepsilon) = \int_{p^{max}:=d^{-1}(0,\varepsilon)}^{p} [v-p]A(d(v,\varepsilon))dd(v,\varepsilon)$. We choose to present our analysis by integrating on the quantity space as it allows us to develop sharper insights given that (i) the allocation function is a mapping from the quantity space and (ii) we can rely on the interpretation of valuations through the inverse demand.

[8] As in section 2, we impose these assumptions for ease of exposition but our results still hold under more general demand functions, such as discontinuity and lack of differentiability in countable disjoint sets.

[9] As mentioned in section 2, the underlying utility function is assumed to be quasilinear and, hence, the demand is only a function of price (and $\boldsymbol{\epsilon}$ in the stochastic case).

[10] The Slutsky conditions are satisfied if the demand cross-derivatives are equal (i.e., $\frac{\partial d_i}{\partial p_j} = \frac{\partial d_j}{\partial p_i}$ for all $i \neq j$).

[11] To simplify notation, we simply write $r^{\boldsymbol{\varepsilon}}$ omitting the arguments $\mathbf{q}$, $\mathbf{p}$, and $\boldsymbol{\epsilon}$.

[12] As in section 2, the integral in Equation (14) can be equivalently expressed by integrating over the price space, namely, $CS^A(\boldsymbol{\varepsilon}) = \int_{C'} \sum_i [s_i^{\boldsymbol{\varepsilon}} - p_i] A_i(d(s^{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon}))dd_i(s^{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon})$, where $s^{\boldsymbol{\varepsilon}} = d^{-1}(r^{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon})$ is the parametric function that corresponds to C' and goes from $\mathbf{p}^{\mathbf{max}} := d^{-1}(\mathbf{0}, \boldsymbol{\varepsilon})$ to $\mathbf{p}$ (this equivalence holds for any allocation rule).

[13] In this case, it suffices to require $\overline{\mathbf{d}} > \mathbf{0}$ and that $\mathbf{B}$ is a symmetric strictly diagonal dominant Z-matrix (i.e., the off-diagonal entries are less or equal than zero).

[14] To ensure that the equilibrium is unique, it is enough to assume that $\sum_{j=1}^{n} B_{ij}$ is greater than $1/[c_i f_i(-A_i)]$, where $f_i(\cdot)$ is the pdf of ϵ_i and A_i is the minimum value in the support of ϵ_i.

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Supporting Information

Additional supporting information may be found online in the Supporting Information section at the end of the article.

Online Appendix.