Online Appendix: Price Discrimination with Fairness Constraints

Appendix A: Proof of Proposition 1 and Theorem 2

Proof of Proposition 1. We refer to Table 2 for convenient formulas used in the proof.

Table 2 Closed-form expressions for linear and exponential demand models.

	$R_i(p)$	$\bar{F}_i(p)$	$S_i(p)$	$N_i(p)$
Linear	$(p-c)\max\{0, 1-\frac{p}{b_i}\}$	$\max\{0, 1-\frac{p}{b_i}\}$	$\frac{(\max\{0, b_i - p\})^2}{2b_i}$	$\frac{\min\{b_i, p\}}{2}$
Exponential	$(p-c)e^{-\lambda_i p}$	$e^{-\lambda_i p}$	$\frac{1}{\lambda_i}e^{-\lambda_i p}$	$\frac{1}{\lambda_i} - \frac{p_i e^{-\lambda_i p_i}}{1-e^{-\lambda_i p_i}}$

(a) We first consider the case of an exponential demand, that is, $V_i \sim Exp(\lambda_i)$ with $\lambda_0 > \lambda_1$. Suppose that we have 1-fairness in price (i.e., there exists a price p such that $p = p_0 = p_1$). Then, we immediately have $\bar{F}_0(p) < \bar{F}_1(p)$, $S_0(p) < S_1(p)$, and $N_0(p) < N_1(p)$, so that 1-fairness in price cannot be satisfied along with another 1-fairness constraint. Similarly, if 1-fairness in demand is satisfied, we have $\bar{F}_0(p_0) = \bar{F}_1(p_1)$, implying that $p_0 = \frac{\lambda_1}{\lambda_0}p_1$. For such prices, we have $S_0(p_0) = \frac{\lambda_1}{\lambda_0}S_1(p_1) < S_1(p_1)$ and $N_1(p_1) - N_0(p_0) = \frac{1}{\lambda_1} - \frac{1}{\lambda_0} + (1-\frac{\lambda_0}{\lambda_1})\frac{p_0 e^{-\lambda_0 p_0}}{1-e^{-\lambda p_0}}$. Note that $f(p) = \frac{1}{\lambda_1} - \frac{1}{\lambda_0} + (1-\frac{\lambda_0}{\lambda_1})\frac{pe^{-\lambda_0 p}}{1-e^{-\lambda p}}$ is a strictly increasing function starting from $f(p) = 0$ and, hence, $N_1(p_1) - N_0(p_0) = f(p_0) > 0$. As a result, 1-fairness in demand cannot coexist with 1-fairness either in surplus or in no-purchase valuation. Finally, satisfying 1-fairness in surplus means that $S_0(p_0) = S_1(p_1)$ and, thus, $p_1 = \frac{\lambda_0}{\lambda_1}p_0 + \frac{1}{\lambda_1}\log\frac{\lambda_0}{\lambda_1}$. We have shown above that $N_1(\frac{\lambda_0}{\lambda_1}p_0) > N_0(p_0)$. Since $N_1(\cdot)$ is an increasing function and $p_1 > \frac{\lambda_0}{\lambda_1}p_0$, we then have $N_1(p_1) > N_0(p_0)$. Consequently, 1-fairness in no-purchase valuation cannot coexist with 1-fairness in surplus. In conclusion, under positive prices, any pair of 1-fairness constraints cannot be simultaneously satisfied.

(b) We next consider the case of a linear demand, that is, $V_i \sim U(0, b_i)$ with $b_0 < b_1$. Suppose that we have 1-fairness in price (i.e., there exists a price p such that $p = p_0 = p_1$). Then, we immediately have $\bar{F}_0(p) < \bar{F}_1(p)$ and $S_0(p) < S_1(p)$, so that 1-fairness in price cannot be satisfied along with another 1-fairness constraint. Similarly, if 1-fairness in demand is satisfied, we have $\bar{F}_0(p_0) = \bar{F}_1(p_1)$. Note the surplus of group i can be written as $b_i(\bar{F}_i(p_i))^2/2$ and, hence, the surplus of group 0 is lower than group 1 under 1-fairness in demand. Finally, note that when both groups have positive demand and positive prices, the no-purchase valuation is always equal to half of the price. Thus, 1-fairness in no-purchase valuation implies 1-fairness in price, so that it cannot coexist with 1-fairness in demand or surplus. This concludes the proof. □

Before presenting the proof of Theorem 2, we first state and prove Lemma 1 which describes necessary and sufficient conditions for $\mathcal{W}'(0)$ to be positive or negative.

Lemma 1. *Suppose that $\bar{F}_i, S_i, N_i$, and R_i are continuous and twice differentiable at p_i^*. Suppose also that $\bar{F}_i$, S_i, and N_i are monotone and invertible.*

(a) W.l.o.g, let $p_0^* < p_1^*$. For price fairness, $\mathcal{W}'(0) > 0$ if and only if $d_1\bar{F}_1(p_1^*)R_0''(p_0^*) - d_0\bar{F}_0(p_0^*)R_1''(p_1^*) < 0$.

(b) W.l.o.g, let $\bar{F}_0(p_0^*) < \bar{F}_1(p_1^*)$. For demand fairness, $\mathcal{W}'(0) > 0$ if and only if $d_1\bar{F}_1(p_1^*)\bar{F}_1'(p_1^*)R_0''(p_0^*) - d_0\bar{F}_0(p_0^*)\bar{F}_0'(p_0^*)R_1''(p_1^*) < 0$.

(c) *W.l.o.g, let $S_0(p_0^*) < S_1(p_1^*)$. For surplus fairness, $\mathcal{W}'(0) > 0$ if and only if $d_1\bar{F}_1(p_1^*)^2 R_0''(p_0^*) - d_0\bar{F}_0(p_0^*)^2 R_1''(p_1^*) > 0$.*

(d) *W.l.o.g, let $N_0(p_0^*) < N_1(p_1^*)$. For no-purchase valuation fairness, $\mathcal{W}'(0) > 0$ if and only if $d_1\bar{F}_1(p_1^*)N_1'(p_1^*)R_0''(p_0^*) - d_0\bar{F}_0(p_0^*)N_0'(p_0^*)R_1''(p_1^*) < 0$.*

Proof of Lemma 1. We discuss the four problems separately.

(a) Price Fairness. Since $\mathcal{W}(\alpha) = R_0(p_0(\alpha)) + R_1(p_1(\alpha)) + d_0 S_0(p_0(\alpha)) + d_1 S_1(p_1(\alpha))$, the derivative of $\mathcal{W}(\alpha)$ is given by:

$$\mathcal{W}'(\alpha) = R_0'(p_0(\alpha))p_0'(\alpha) + R_1'(p_1(\alpha))p_1'(\alpha) + d_0 S_0'(p_0(\alpha))p_0'(\alpha) + d_1 S_1'(p_1(\alpha))p_1'(\alpha).$$

By definition, at $\alpha = 0$ we have $p_i(0) = p_i^*$ and $R_i'(p_i(0)) = 0$. Thus, we obtain $\mathcal{W}'(0) = d_0 S_0'(p_0^*)p_0'(0) + d_1 S_1'(p_1^*)p_1'(0)$. By definition of the normalized surplus function $S(\cdot)$, $S_i'(p) = -\bar{F}_i(p)$ and thus we have $\mathcal{W}'(0) = -d_1\bar{F}_1(p_1^*)p_1'(0) - d_0\bar{F}_0(p_0^*)p_0'(0)$. The rest of the proof relies on computing $p_i'(0)$ for each fairness definition, which we shall do in cases.

For price fairness, since we assume that $p_0^* < p_1^*$, the seller has to increase p_0 and decrease p_1 in order to improve price fairness. Let $\Delta p_0(\alpha) = p_0(\alpha) - p_0^*$, and $\Delta p_1(\alpha) = p_1^* - p_1(\alpha)$. Hence, $p_0'(0) = \lim_{\alpha\to 0} \Delta p_0(\alpha)/\alpha$, and $p_1'(0) = \lim_{\alpha\to 0} -\Delta p_1(\alpha)/\alpha$. Given α, the profit optimization problem (1) for the seller can be cast as

$$\begin{aligned} \max\ & R_0(p_0^* + \Delta p_0) + R_1(p_1^* - \Delta p_1) && (4)\\ \text{s.t. } & \Delta p_0 + \Delta p_1 \geq (p_1^* - p_0^*)\alpha && (5)\\ & \Delta p_0, \Delta p_1 \geq 0, \end{aligned}$$

where Eq. (5) requires that the total price changes is at least $(p_1^* - p_0^*)\alpha$. Further, the profit objective (4) can be expanded using a Taylor expansion around (p_0^*, p_1^*) as

$$R_0(p_0^*) + R_0'(p_0^*)\Delta p_0 + \frac{1}{2}R_0''(p_0^*)\Delta p_0^2 + g_0(\Delta p_0) + R_1(p_1^*) - R_1'(p_1^*)\Delta p_1 + \frac{1}{2}R_1''(p_1^*)\Delta p_1^2 + g_1(\Delta p_1), \quad (6)$$

where $g_i(\Delta p_i)$ corresponds to the remainder term. Since R_i is twice differentiable, g_i must be twice differentiable and $g_i''(0) = 0$ since $R_i''(p_0^*) = g_i''(0)$. Removing the constants $R_i(p_i^*)$ from (6) and recalling that $R_i'(p_i^*) = 0$, we can rewrite the profit optimization problem as

$$\begin{aligned} \min\ & -\frac{1}{2}R_0''(p_0^*)\Delta p_0^2 - \frac{1}{2}R_1''(p_1^*)\Delta p_1^2 + g_0(\Delta p_0) + g_1(\Delta p_1) && \text{[Minimize the profit loss]} \quad (7)\\ \text{s.t. } & \Delta p_0 + \Delta p_1 \geq (p_1^* - p_0^*)\alpha \\ & \Delta p_i \geq 0. \end{aligned}$$

The KKT conditions for Eq. (7) are given by:

$$\begin{aligned} \begin{bmatrix} -R_0''(p_0^*)\Delta p_0 + g_0'(\Delta p_0) \\ -R_1''(p_1^*)\Delta p_1 + g_1'(\Delta p_1) \end{bmatrix} &= \mu \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \\ \Delta p_0 + \Delta p_1 &\geq (p_1^* - p_0^*)\alpha, \\ \Delta p_i &\geq 0, \\ \mu &\geq 0, \\ \mu\left[(p_1^* - p_0^*)\alpha - \Delta p_0 - \Delta p_1\right] &= 0. \end{aligned}$$

This can be further reduced to

$$-R_0''(p_0^*)\Delta p_0 + g_0'(\Delta p_0) = -R_1''(p_1^*)((p_1^* - p_0^*)\alpha - \Delta p_0) + g_1'((p_1^* - p_0^*)\alpha - \Delta p_0), \tag{8}$$

$$\Delta p_0 \in [0, (p_1^* - p_0^*)\alpha]. \tag{9}$$

Since we assume that R_i is twice differentiable, $R_i''(p_i) = R_i''(p_i^*) + g_i''(\Delta p_i)$ is well defined and

$$\lim_{\alpha\to 0} g_i'(\Delta p_i)/\alpha = \lim_{\alpha\to 0} \frac{g_i'(\Delta p_i)}{\Delta p_i}\frac{\Delta p_i}{\alpha} = \lim_{\alpha\to 0} g''(\Delta p_i)\frac{\Delta p_i}{\alpha} = 0,$$

where the last equality comes from the facts that $g_i''(0) = 0$ and that $\Delta p_i/\alpha$ is bounded from Eq. (9). Thus, by dividing both sides of Eq. (8) by α and taking the limit as α goes to 0, we obtain:

$$-R_0''(p_0^*)p_0'(0) = -R_1''(p_1^*)[p_1^* - p_0^* - p_0'(0)]. \tag{10}$$

As a result of Eq. (10), as α goes to 0, we have the expression of $p_0'(0)$ and $p_1'(0)$ (with a similar argument):

$$p_0'(0) = \frac{R_1''(p_1^*)}{R_0''(p_0^*) + R_1''(p_1^*)}(p_1^* - p_0^*) \text{ and } p_1'(0) = -\frac{R_0''(p_0^*)}{R_0''(p_0^*) + R_1''(p_1^*)}(p_1^* - p_0^*). \tag{11}$$

Recall that we require $\mathcal{W}'(0) = -d_1\bar{F}_1(p_1^*)p_1'(\alpha) - d_0\bar{F}_0(p_0^*)p_0'(\alpha) > 0$. By substituting Eq. (11) into the previous equation, we obtain our desired result $d_1\bar{F}_1(p_1^*)R_0''(p_0^*) - d_0\bar{F}_0(p_0^*)R_1''(p_1^*) < 0$.

(b) Demand Fairness. For demand fairness, since we assume that group 1 has higher demand, then p_0 decreases and p_1 increases. Note that the objective function is the same as Eq. (4), whereas Eq. (5) becomes

$$\bar{F}_0(p_0^* - \Delta p_0) - \bar{F}_0(p_0^*) + \bar{F}_1(p_1^*) - \bar{F}_1(p_1^* + \Delta p_1) \geq \alpha[\bar{F}_1(p_1^*) - \bar{F}_0(p_0^*)].$$

Writing the demand change into Taylor expansion, we have

$$-\bar{F}_0'(p_0^*)\Delta p_0 - \bar{F}_1'(p_1^*)\Delta p_1 + h_0(\Delta p_0) + h_1(\Delta p_1) \geq \alpha[\bar{F}_1(p_1^*) - \bar{F}_0(p_0^*)],$$

where $h_i(\Delta p_i)$ is the remainder term in demand. Since the demand is differentiable, we know that h_i' is well defined and $h_i'(0) = 0$, as $\bar{F}_i'(p_i^*) = \bar{F}_i'(p_i^*) + h_i'(0)$. We setup an optimization problem as in Eq. (7), and the KKT conditions for the new problem are given by:

$$\begin{aligned}
\begin{bmatrix} -R_0''(p_0^*)\Delta p_0 + g_0'(\Delta p_0) \\ -R_1''(p_1^*)\Delta p_1 + g_1'(\Delta p_1) \end{bmatrix} &= \mu \begin{bmatrix} \bar{F}_0'(p_0^*) - h_0'(\Delta p_0) \\ \bar{F}_1'(p_1^*) - h_1'(\Delta p_1) \end{bmatrix}, \\
-\bar{F}_0'(p_0^*)\Delta p_0 - \bar{F}_1'(p_1^*)\Delta p_1 + h_0(\Delta p_0) + h_1(\Delta p_1) &\geq \alpha[\bar{F}_1(p_1^*) - \bar{F}_0(p_0^*)], \\
\Delta p_i &\geq 0, \\
\mu &\geq 0, \\
\mu\left[(p_1^* - p_0^*)\alpha - \Delta p_0 - \Delta p_1\right] &= 0.
\end{aligned}$$

This can be further reduced to

$$-R_0''(p_0^*)\Delta p_0 + g_0'(\Delta p_0) = \frac{\bar{F}_0'(p_0^*) - h_0'(\Delta p_0)}{\bar{F}_1'(p_1^*) - h_1'(\Delta p_1)}\left[-R_1''(p_1^*)\Delta p_1 + g_1'(\Delta p_1)\right], \tag{12}$$

$$-\bar{F}_0'(p_0^*)\Delta p_0 - \bar{F}_1'(p_1^*)\Delta p_1 + h_0(\Delta p_0) + h_1(\Delta p_1) = \alpha[\bar{F}_1(p_1^*) - \bar{F}_0(p_0^*)]. \tag{13}$$

Since $\bar{F}_i(p)$ is twice differentiable, $h_i'(\Delta p_i)$ is well defined and $\lim_{\Delta p_i \to 0} h_i'(\Delta p_i) = 0$. If $p_i'(0)$ is bounded (as we will show later), then $\lim_{\alpha\to 0} h_i(\Delta p_i)/\alpha = \lim_{\alpha \to 0} \frac{h_i(\Delta p_i)}{\Delta p_i}\frac{\Delta p_i}{\alpha} = 0$. Similarly, we have $\lim_{\alpha\to 0} g_i'(\Delta p_i)/\alpha = 0$. Thus, dividing Eq. (12) and Eq. (13) by α and taking the limit as α goes to 0, leads to

$$-R_0''(p_0^*)[-p_0'(0)] = \frac{\bar{F}_0'(p_0^*)}{\bar{F}_1'(p_1^*)}\left[-R_1''(p_1^*)p_1'(0)\right],$$

$$-\bar{F}_0'(p_0^*)[-p_0'(0)] - \bar{F}_1'(p_1^*)p_1'(0) = [\bar{F}_1(p_1^*) - \bar{F}_0(p_0^*)].$$

Note that as opposed to price fairness, we now have $\lim_{\alpha\to 0} \Delta p_0(\alpha)/\alpha = -p_0'(0)$ and $\lim_{\alpha\to 0}\Delta p_1(\alpha)/\alpha = p_1'(0)$ (i.e., the sign is reversed), as p_0 decreases and p_1 increases with α. Hence, we have:

$$p_0'(0) = \frac{R_1''(p_1^*)\bar{F}_0'(p_0^*)}{R_0''(p_0^*)\bar{F}_1'(p_0^*)^2 + R_1''(p_1^*)\bar{F}_1'(p_0^*)^2}\Delta w \text{ and } p_1(0)' = -\frac{R_0''(p_0^*)\bar{F}_1'(p_1^*)}{R_0''(p_0^*)\bar{F}_1'(p_0^*)^2 + R_1''(p_1^*)\bar{F}_1'(p_0^*)^2}\Delta w,$$

where $\Delta w = [\bar{F}_1(p_1^*) - \bar{F}_0(p_0^*)]$. Again, we require that $-d_0\bar{F}_0(p_0^*)p_0'(0) - d_1\bar{F}_1(p_1^*)p_1'(0) > 0$. Following the same line of argument as for price fairness, we obtain:

$$\mathcal{W}'(0) = -d_0\bar{F}_0(p_0^*)p_0'(0) - d_1\bar{F}_1(p_1^*)p_1'(0) > 0 \iff d_1\bar{F}_1(p_1^*)\bar{F}_1'(p_1^*)R_0''(p_0^*) - d_0\bar{F}_0(p_0^*)\bar{F}_0'(p_0^*)R_1''(p_1^*) < 0.$$

We next show that $p_i'(0)$ is indeed bounded. Consider p_0 as an example. Since the demand change in group 1 is non-negative, from Eq. (13) we have $-\bar{F}_0'(p_0^*)\Delta p_0 + h_0(\Delta p_0) \le \alpha[\bar{F}_1(p_1^*) - \bar{F}_0(p_0^*)]$. Let $\Delta\bar{p}_0 = p_0^* - \bar{F}_0^{-1}(\bar{F}_0(p_0^*) + \alpha[\bar{F}_1(p_1^*) - \bar{F}_0(p_0^*)])$, i.e., $\Delta\bar{p_0}$ increases the demand by $\alpha[\bar{F}_1(p_1^*) - \bar{F}_0(p_0^*)]$. Since the demand is monotone, $\Delta p_0 \le \Delta\bar{p_0}$. We then have:

$$\begin{aligned}
|p_0'(0)| = \lim_{\alpha\to 0}\frac{\Delta p_0}{\alpha} &\le \lim_{\alpha\to 0}\frac{\Delta\bar{p_0}}{\alpha}\\
&= \lim_{\alpha\to 0}\frac{\bar{F}_0^{-1}(F_0(p_0^*)) - \bar{F}_0^{-1}(\bar{F}_0(p_0^*) + \alpha[\bar{F}_1(p_1^*) - \bar{F}_0(p_0^*)])}{\alpha}\\
&= -[\bar{F}_1(p_1^*) - \bar{F}_0(p_0^*)]\cdot(\bar{F}_0^{-1})'[\bar{F}_0(p_0^*)]\\
&= -[\bar{F}_1(p_1^*) - \bar{F}_0(p_0^*)]\frac{1}{\bar{F}_0'(p_0^*)}.
\end{aligned}$$

Hence, showing that $p_i'(0)$ is bounded.

The proof for (c) surplus fairness follows a similar argument as in (b). The proof for (d) no-purchase valuation fairness is also similar to (b), but note that in this case, p_0 increases and p_1 decreases, so that the sign of $p_i'(0)$ is reversed. □

Proof of Theorem 2. For linear demand, without loss of generality, we assume that $V_i \sim U(0, b_i)$, with $b_0 < b_1$. For exponential demand, without loss of generality, we assume that $V_i \sim Exp(\lambda_i)$, with $\lambda_0 > \lambda_1$. See Table 2 for the closed form expressions of $R_i, \bar{F}_i, S_i, N_i$. One can check that in both cases, we have $p_0^* < p_1^*$, $\bar{F}_0(p_0^*) < \bar{F}_1(p_1^*)$, $S_0(p_0^*) < S_1(p_1^*)$, and $N_0(p_0^*) < N_1(p_1^*)$. We report the expressions of $p_i^*, R_i''(p_i^*), \bar{F}_i'(p_i^*), \bar{F}_i(p_i^*)$, and $N_i'(p_i^*)$ in Table 3. By substituting these expressions into the conditions in Lemma 1, one can show that the inequalities for price and no-purchase valuation fairness are always satisfied, whereas the inequalities for demand and surplus fairness conditions are always violated. □

Table 3 Function values for linear and exponential demand models.

	p_i^*	$R_i''(p_i^*)$	$\bar{F}_i'(p_i^*)$	$\bar{F}_i(p_i^*)$	$N_i'(p_i^*)$
Linear	$\frac{b_i+c}{2}$	$-2d_i/b_i$	$-1/b_i$	$\frac{b_i-c}{2}$	$\frac{1}{2}$
Exponential	$\frac{1}{\lambda_i}+c$	$-\lambda_i e^{-1-\lambda_i c}$	$-\lambda_i e^{-1-\lambda_i c}$	$e^{-1-\lambda_i c}$	$\frac{\lambda_i c e^{1+\lambda c}+1}{(e^{1+\lambda c}-1)^2}$

Appendix B: Proofs of Propositions 2 to 5

In optimization problem (1), we may use prices which are not in $[0,b_i]$, so that the problem is not necessarily convex. To make the analysis simpler, in each proposition we discuss four cases separately: (1) $p_i \le b_i$ for both groups; (2) $p_0 > b_0$ and $p_1 \le b_i$; (3) $p_0 \le b_0$ and $p_1 > b_1$; and (4) $p_i > b_i$ for both groups. We then compare the optimal solution for each case and characterize the optimal solution to the problem. We first point out that case (4) can be eliminated as it leads to zero profit, and we do not discuss this case in the subsequent proofs. Note that for case (1), for each fairness metric $M_i(p_i)$, we always have $M_0(p_0^*) \le M_0(p_0) \le M_1(p_1) \le M_1(p_1^*)$. First, note that if $M_i(p_i)$ is not in $[M_0(p_0^*), M_1(p_1^*)]$, we can set the price of group i to be p_i^*, such that the solution is still feasible, but the profit is higher because for group i we use the unconstrained optimal price. Second, if $M_1(p_1) < M_0(p_0)$, then one can construct another solution p_i' such that $M_0(p_0') = M_1(p_1)$ and $M_1(p_1') = M_0(p_0)$, which is also a feasible solution. However, because $M_i(p_i')$ is closer to $M_i(p_i^*)$, the price changes less when compared to using p_i, and hence the constructed prices have higher profit. Finally, in case (1) we have w.l.o.g. that $M_1(p_1) - M_0(p_0) = (1-\alpha)|M_1(p_1^*) - M_0(p_0^*)|$, i.e., the solution is tight. If this is not the case, one can fix $M_0(p_0)$ and increase $M_1(p_1)$ slightly such that the solution is still feasible. However, by moving $M_1(p_1)$ closer to the unconstrained level, the profit can only increase.

Proof of Proposition 2. Here we first expand the proposition in the main body with closed-form solution for prices, as well as changes in profit, consumer surplus and social welfare. Let $w = p_1^* - p_0^* = \frac{b_1-b_0}{2}$. If $0 \le \alpha \le \tilde{\alpha}_p$, then

$$p_0(\alpha) = p_0^* + \frac{d_1 b_0}{d_0 b_1 + d_1 b_0}\alpha w \quad \text{and} \quad p_1(\alpha) = p_1^* - \frac{d_0 b_1}{d_0 b_1 + d_1 b_0}\alpha w.$$

The changes in profit, consumer surplus, and social welfare are given by:

$$\begin{aligned}
\mathcal{R}(\alpha) - \mathcal{R}(0) &= -\frac{d_0 d_1}{d_0 b_1 + d_1 b_0}(\alpha w)^2 \le 0,\\
\mathcal{S}(\alpha) - \mathcal{S}(0) &= \frac{d_0 d_1}{2(d_0 b_1 + d_1 b_0)}\left[(b_1 - b_0)\alpha w + (\alpha w)^2\right] \ge 0,\\
\mathcal{W}(\alpha) - \mathcal{W}(0) &= \frac{d_0 d_1}{2(d_0 b_1 + d_1 b_0)}\left[(b_1 - b_0)\alpha w - (\alpha w)^2\right] \ge 0.
\end{aligned}$$

If $\tilde{\alpha}_p < \alpha \le 1$, then

$$p_0(\alpha) = p_1(\alpha) = p_1^* = \frac{b_1 + c}{2} > b_0$$

and

$$\begin{aligned}
\mathcal{R}(\alpha) - \mathcal{R}(0) &= -R_0(p_0^*) < 0,\\
\mathcal{S}(\alpha) - \mathcal{S}(0) &= -d_0 S_0(p_0^*) < 0,\\
\mathcal{W}(\alpha) - \mathcal{W}(0) &= -R_0(p_0^*) - d_0 S_0(p_0^*) < 0.
\end{aligned}$$

We next prove these statements below.

Now we prove the above statements by analyzing the three possible cases. For case (3), the price difference is greater than $b_1 - b_0$. Since $p_1^* - p_0^* = (b_1 - b_0)/2$, then any price policy for case (3) is infeasible. We next analyze the profit from cases (1) and (2).

Case (1): Let Δp_0 and Δp_1 be the price changes, that is, $p_0 = p_0^* + \Delta p_0$ and $p_1 = p_1^* - \Delta p_1$. Let $w = (b_1 - b_0)/2$. For the seller, the profit optimization problem in Eq. (7) can be written as:

$$\begin{aligned} \min\ & \frac{d_0}{b_0}\Delta p_0^2 + \frac{d_1}{b_1}\Delta p_0^2 \\ \text{s.t. } & \Delta p_0 + \Delta p_1 = \alpha w \\ & \Delta p_0 \le b_0 - p_0^* \\ & \Delta p_0 \le p_1^* \\ & \Delta p_i \ge 0. \end{aligned}$$

We first relax the upper-bound constraints, and then characterize the condition under which such constraints are not tight. When the upper-bound constraints are removed, solving the above problem leads to

$$\Delta p_0 = \frac{d_1 b_0}{d_0 b_1 + d_1 b_0}\alpha w \text{ and } \Delta p_1 = \frac{d_0 b_1}{d_0 b_1 + d_1 b_0}\alpha w. \tag{14}$$

By substituting p_0 and p_1 into the profit, consumer surplus, and social welfare functions, we obtain:

$$\begin{aligned} \mathcal{R}(\alpha) - \mathcal{R}(0) &= -\frac{d_0 d_1}{d_0 b_1 + d_1 b_0}(\alpha w)^2, \\ \mathcal{S}(\alpha) - \mathcal{S}(0) &= \frac{d_0 d_1}{2(d_0 b_1 + d_1 b_0)}\left[(b_1 - b_0)\alpha w + (\alpha w)^2\right], \\ \mathcal{W}(\alpha) - \mathcal{W}(0) &= \frac{d_0 d_1}{2(d_0 b_1 + d_1 b_0)}\left[(b_1 - b_0)\alpha w - (\alpha w)^2\right]. \end{aligned}$$

Such a solution is valid as long as Δp_i does not reach the upper bounds. Specifically, taking Eq. (14) into $\Delta p_0 \le b_0 - p_0^*$, we have $\alpha \le \frac{d_0 b_1 + d_1 b_0}{d_1 b_0}\frac{b_0 - c}{b_1 - b_0}$. On the other hand, $\Delta p_1 \le p_1^*$ implies that $\alpha \le \frac{d_0 b_1 + d_1 b_0}{d_0 b_1}\frac{b_1 + c}{b_1 - b_0}$, which always holds since the right-hand side is greater than 1. We will later argue that we do not need to consider the case when $\alpha > \frac{d_0 b_1 + d_1 b_0}{d_1 b_0}\frac{b_0 - c}{b_1 - b_0}$.

Case (2): In this case, $p_0 > b_0$, so that both the profit and the consumer surplus from group 0 are zero. For group 1, the optimal price is p_1^*. Therefore, the optimal solution is always $p_0 = p_1 = p_1^*$, the profit loss is $R_0(p_0^*)$, and the consumer surplus loss is $S_0(p_0^*)$.

We next compare cases (1) and (2). First, the analysis of case (1) is only valid when $p_0 + \Delta p_0 \le b_0$, i.e., $\alpha \le \frac{d_0 b_1 + d_1 b_0}{d_1 b_0}\frac{b_0 - c}{b_1 - b_0}$. On the other hand, by comparing the profit for cases (1) and (2), one can see that for small values of α, the profit in case (1) is larger (close to optimal), whereas the profit in case (2) is fixed. If α is large enough, the profit loss in case (1) is greater than the profit from group 0. Formally,

$$\frac{d_0 d_1}{d_0 b_1 + d_1 b_0}(\alpha w)^2 \ge \frac{d_0 (b_0 - c)^2}{4 b_0}.$$

By rearranging terms, we obtain $\alpha \ge \sqrt{\frac{d_0 b_1 + d_1 b_0}{d_1 b_0}}\frac{b_0 - c}{b_1 - b_0}$. Thus, when $\alpha \le \sqrt{\frac{d_0 b_1 + d_1 b_0}{d_1 b_0}}\frac{b_0 - c}{b_1 - b_0}$, case (1) has a higher profit. The transition from case (1) to (2) happens either when case (1) is not feasible (i.e., $\alpha > \frac{d_0 b_1 + d_1 b_0}{d_1 b_0}\frac{b_0 - c}{b_1 - b_0}$), or when it has a lower profit (i.e., $\alpha > \sqrt{\frac{d_0 b_1 + d_1 b_0}{d_1 b_0}}\frac{b_0 - c}{b_1 - b_0}$). Since $\sqrt{\frac{d_0 b_1 + d_1 b_0}{d_1 b_0}}\frac{b_0 - c}{b_1 - b_0} \le \frac{d_0 b_1 + d_1 b_0}{d_1 b_0}\frac{b_0 - c}{b_1 - b_0}$, then the transition happens before p_0 reaches b_0. Consequently, the threshold value is $\tilde{\alpha}_p = \max\left(\sqrt{\frac{d_0 b_1 + d_1 b_0}{d_1 b_0}}\frac{b_0 - c}{b_1 - b_0}, 1\right)$. Accordingly, $\mathcal{W}(\alpha)$ first increases with α, but after $\tilde{\alpha}$, $\mathcal{W}(\alpha)$ "jumps" below $\mathcal{W}(0)$. $\square$

Proof of Proposition 3. Similar to price fairness, we first provide the closed-form solution with respect to α. For demand fairness, let $w=\frac{(b_1-b_0)c}{2b_0b_1}$. Then, we have:

$$p_0(\alpha)=p_0^*-\frac{d_1b_0b_1}{d_0b_1+d_1b_0}\alpha w \quad \text{and} \quad p_1(\alpha)=p_1^*+\frac{d_0b_0b_1}{d_0b_0+d_1b_1}\alpha w.$$

The changes in profit, consumer surplus, and social welfare are given by:

$$\begin{aligned}
\mathcal{R}(\alpha)-\mathcal{R}(0)&=-\frac{d_0d_1b_0b_1}{d_0b_0+d_1b_1}(\alpha w)^2\leq 0,\\
\mathcal{S}(\alpha)-\mathcal{S}(0)&=\frac{d_0d_1}{2(d_0b_0+d_1b_1)}\Big[-(b_1-b_0)c\alpha w+b_0b_1(\alpha w)^2\Big]\leq 0,\\
\mathcal{W}(\alpha)-\mathcal{W}(0)&=\frac{d_0d_1}{2(d_0b_0+d_1b_1)}\Big[-(b_1-b_0)c\alpha w-b_0b_1(\alpha w)^2\Big]\leq 0.
\end{aligned}$$

We next prove these statements below.

We first consider case (1), where $p_i\leq b_i$ for both groups. Let Δp_0 and Δp_1 be the price changes, that is, $p_0=p_0^*-\Delta p_0$ and $p_1=p_1^*+\Delta p_1$. The initial difference in demand is $(b_1-b_0)c/2b_0b_1$. Similar to Proposition 2, the optimization problem is given by:

$$\begin{aligned}
\min\ & \frac{d_0}{b_0}\Delta p_0^2+\frac{d_1}{b_1}\Delta p_0^2\\
\text{s.t.}\ & \frac{\Delta p_0}{b_0}+\frac{\Delta p_1}{b_1}=\alpha\frac{(b_1-b_0)c}{2b_0b_1}\\
& \Delta p_0\leq p_0^*\\
& \Delta p_0\leq b_1-p_1^*\\
& \Delta p_i\geq 0.
\end{aligned}$$

If we ignore the upper bounds, the above problem leads to

$$\Delta p_0=\frac{d_1b_0b_1}{d_0b_0+d_1b_1}\alpha w \text{ and } \Delta p_1=\frac{d_0b_0b_1}{d_0b_0+d_1b_1}\alpha w,$$

where $w=\frac{(b_1-b_0)c}{2b_0b_1}$. Substituting p_0 and p_1 into the profit, consumer surplus, and social welfare (defined in Section 3) yields the desired result. Note that the above analysis holds for any α, as the prices will not reach either boundary (0 and b_i). This follows from the fact that p_0 decreases and p_1 increases with α, and the demand can be matched before one of the prices reaches the boundary.

For case (2), one can observe that the demand fairness and profit are not impacted whether $p_0>b_0$ or is exactly $p_0=b_0$. Thus, without loss of generality case (2) is subsumed by case (1). For a similar reason, case (3) is subsumed by case (1).

It is easy to see that $\mathcal{R}(\alpha)-\mathcal{R}(0)$ is always negative. To see that $\mathcal{S}(\alpha)-\mathcal{S}(0)$ is always negative, note that $\mathcal{S}(\alpha)-\mathcal{S}(0)=0$ at $\alpha=0$ and decreases with α on $\alpha\in[0,\frac{(b_1-b_0)c}{2b_0b_1w}]$, where $\frac{(b_1-b_0)c}{2b_0b_1w}=1$. ☐

Proof of Proposition 4. For case (2), one can observe that the surplus fairness and profit are not impacted if $p_0>b_0$ or is exactly $p_0=b_0$. Thus, without loss of generality case (2) is subsumed by case (1). For a similar reason, case (3) is subsumed by case (1). Thus, we only need to consider case (1), where $p_i\leq b_i$ for both groups.

For case (1), the normalized consumer surplus is given by $S_i(p_i)=\frac{(b_i-p)(1-p/b_i)}{2}$. Correspondingly, when we use $p_0=p_0^*-\Delta p_0$ and $p_1=p_1^*+\Delta p_1$, we have $S_0(p_0^*-\Delta p_0)-S_0(p_0^*)=\frac{b_0-c}{2b_0}\Delta p_0+\frac{1}{2b_0}\Delta p_0^2$ and $S_1(p_1^*)-$

$S_1(p_1^*+\Delta p_1)=\frac{b_1-c}{2b_1}\Delta p_1-\frac{1}{2b_1}\Delta p_1^2$. The initial consumer surplus difference is $\frac{(b_1-c)^2}{8b_1}-\frac{(b_0-c)^2}{8b_0}$. Hence, the fairness constraint is given by

$$\frac{b_0-c}{2b_0}\Delta p_0+\frac{1}{2b_0}\Delta p_0^2+\frac{b_1-c}{2b_1}\Delta p_1-\frac{1}{2b_1}\Delta p_1^2=\alpha\left[\frac{(b_1-c)^2}{8b_1}-\frac{(b_0-c)^2}{8b_0}\right].$$

The optimization problem now becomes:

$$\begin{aligned}\min\ & \frac{d_0}{b_0}\Delta p_0^2+\frac{d_1}{b_1}\Delta p_0^2\\ \text{s.t.}\ & \frac{b_0-c}{2b_0}\Delta p_0+\frac{1}{2b_0}\Delta p_0^2+\frac{b_1-c}{2b_1}\Delta p_1-\frac{1}{2b_1}\Delta p_1^2=\alpha\left[\frac{(b_1-c)^2}{8b_1}-\frac{(b_0-c)^2}{8b_0}\right]\\ & \Delta p_0\le p_0^*\\ & \Delta p_0\le b_1-p_1^*\\ & \Delta p_i\ge 0.\end{aligned}$$

We consider the two cases separately: (a) α is small such that $\Delta p_0<p_0^*$, and (b) α is large so that $\Delta p_0=p_0^*$. For case (a), we relax the upper-bound constraints, leading to the following KKT conditions:

$$\begin{gathered}\begin{bmatrix}\frac{2d_0}{b_0}\Delta p_0\\ \frac{2d_1}{b_1}\Delta p_1\end{bmatrix}=\lambda\begin{bmatrix}\frac{b_0-c}{2b_0}+\frac{1}{b_0}\Delta p_0\\ \frac{b_1-c}{2b_1}-\frac{1}{b_1}\Delta p_1\end{bmatrix},\\ \frac{b_0-c}{2b_0}\Delta p_0+\frac{1}{2b_0}\Delta p_0^2+\frac{b_1-c}{2b_1}\Delta p_1-\frac{1}{2b_1}\Delta p_1^2=\alpha\left[\frac{(b_1-c)^2}{8b_1}-\frac{(b_0-c)^2}{8b_0}\right],\\ \Delta p_i\ge 0.\end{gathered}\tag{15}$$

Consequently, we have

$$\Delta p_0=\frac{(b_0-c)\lambda}{4d_0-2\lambda}\text{ and }\Delta p_1=\frac{(b_1-c)\lambda}{4d_1+2\lambda},\tag{16}$$

where λ satisfies

$$\frac{b_0-c}{2b_0}\frac{(b_0-c)\lambda}{4d_0-2\lambda}+\frac{1}{2b_0}\left[\frac{(b_0-c)\lambda}{4d_0-2\lambda}\right]^2+\frac{b_1-c}{2b_1}\frac{(b_1-c)\lambda}{4d_1+2\lambda}-\frac{1}{2b_1}\left[\frac{(b_1-c)\lambda}{4d_1+2\lambda}\right]^2=\alpha\left[\frac{(b_1-c)^2}{8b_1}-\frac{(b_0-c)^2}{8b_0}\right].\tag{17}$$

While the closed-form expression of λ can be computed by transforming Eq. (17) into a quartic function, it is complicated and we do not necessarily need it. On the other hand, we use several properties of λ. First, λ must always be positive due to the fact that $2\frac{d_0}{b_0}\Delta p_0=\lambda(\frac{b_0-c}{2b_0}+\frac{1}{b_0})\Delta p_0$. Second, both Δp_0 and Δp_1 increase with λ, for $\lambda\in[0,2d_0]$, and since $\Delta p_0\ge 0$, the case with $\lambda>2d_0$ never occurs. Third, note that the left-hand side of Eq. (15) increases with both Δp_0 and Δp_1, for $\Delta p_0\in[0,p_0^*]$ and $\Delta p_1\in[0,b_1-p_1^*]$, while the right-hand side increases with α. Together with the fact that Δp_0 and Δp_1 increase with λ, we know that λ increases with α, and one can check that $\lambda=0$ when $\alpha=0$.

We next show that $\mathcal{W}(\alpha)<\mathcal{W}(0)$ for $\alpha\in(0,1]$. Recall that the profit loss is $\frac{d_0}{b_0}\Delta p_0^2+\frac{d_1}{b_1}\Delta p_1^2$ and the surplus change is $d_0\frac{b_0-c}{2b_0}\Delta p_0+\frac{d_0}{2b_0}\Delta p_0^2-d_1\frac{b_1-c}{2b_1}\Delta p_1+\frac{d_1}{2b_1}\Delta p_1^2$. Thus, the social welfare change is $d_0\frac{b_0-c}{2b_0}\Delta p_0-\frac{d_0}{2b_0}\Delta p_0^2-d_1\frac{b_1-c}{2b_1}\Delta p_1-\frac{d_1}{2b_1}\Delta p_1^2$. Since $-\frac{d_0}{2b_0}\Delta p_0^2-\frac{d_1}{2b_1}\Delta p_1^2$ is negative for any α, we only need to show that $d_0\frac{b_0-c}{2b_0}\Delta p_0-d_1\frac{b_1-c}{2b_1}\Delta p_1$ is negative for any α. By substituting Eq. (16) into $d_0\frac{b_0-c}{2b_0}\Delta p_0-d_1\frac{b_1-c}{2b_1}\Delta p_1$, we obtain:

$$d_0\frac{b_0-c}{2b_0}\Delta p_0-d_1\frac{b_1-c}{2b_1}\Delta p_1=\frac{\left[d_0b_1(b_0-c)^2+d_1b_0(b_1-c)^2\right]\lambda^2+2d_0d_1\left[b_1(b_0-c)^2-b_0(b_1-c)^2\right]\lambda}{2b_0b_1(2d_0-\lambda)(2d_1+\lambda)}.\tag{18}$$

The denominator of Eq. (18) is positive since $\Delta p_i\ i=0,1$ are positive. The numerator of Eq. (18) is negative for $\lambda\in(0,\bar{\lambda})$, where

$$\bar{\lambda}=\frac{2d_0d_1\left[b_0(b_1-c)^2-b_1(b_0-c)^2\right]}{d_0b_1(b_0-c)^2+d_1b_0(b_1-c)^2}.$$

We claim that the largest possible value of λ is below $\bar{\lambda}$, so that $d_0\frac{b_0-c}{2b_0}\Delta p_0-d_1\frac{b_1-c}{2b_1}\Delta p_1$ is negative for any α. To show this claim, we substitute $\bar{\lambda}$ into the left-hand side of Eq. (17) and obtain:

$$\frac{b_0-c}{2b_0}\frac{(b_0-c)\bar{\lambda}}{4d_0-2\bar{\lambda}}+\frac{1}{2b_0}\left[\frac{(b_0-c)\bar{\lambda}}{4d_0-2\bar{\lambda}}\right]^2+\frac{b_1-c}{2b_1}\frac{(b_1-c)\bar{\lambda}}{4d_1+2\bar{\lambda}}-\frac{1}{2b_1}\left[\frac{(b_1-c)\bar{\lambda}}{4d_1+2\bar{\lambda}}\right]^2-\left[\frac{(b_1-c)^2}{8b_1}-\frac{(b_0-c)^2}{8b_0}\right]$$
$$=\frac{\left\{b_0^2b_1d_0+b_1c^2d_0+b_0\left[b_1^2d_1+c^2d_1-2b_1c(d_0+d_1)\right]\right\}^2(b_1-b_0)(b_1b_0-c^2)}{8b_0b_1(b_0-c)^2(b_1-c)^2(d_0+d_1)^2}>0,$$

where the inequality comes from the facts that $b_1-b_0>0$ and $b_1b_0-c^2>0$. This indicates that if λ reaches $\bar{\lambda}$, the surplus change is greater than $\left[\frac{(b_1-c)^2}{8b_1}-\frac{(b_0-c)^2}{8b_0}\right]$, which is equal to the initial difference. As a result, λ will never reach $\bar{\lambda}$, and the corresponding $\mathcal{W}(\alpha)-\mathcal{W}(0)$ is always negative.

The above analysis holds only for case (a), that is, before p_0 reaches zero. For case (b), if there exists $\tilde{\alpha}$ such that $\Delta p_0(\tilde{\alpha})=p_0^*$, then for $\alpha>\tilde{\alpha}$, p_0 stays at zero and p_1 increases monotonically. In this case, $\mathcal{W}(\alpha)$ decreases with α, and $\mathcal{W}(\alpha)<\mathcal{W}(\tilde{\alpha})<\mathcal{W}(0)$. □

Proof of Proposition 5. Similar to price fairness, we first provide closed-form solution to the problem. For no-purchase valuation fairness, let $\tilde{\alpha}_n=\min\left(\frac{d_1b_0+d_0b_1}{d_1b_0}\frac{b_0-c}{b_1-b_0},1\right)$ and $w=p_1^*-p_0^*=(b_1-b_0)/2$. If $\alpha\le\tilde{\alpha}_n$, then

$$p_0(\alpha)=p_0^*+\frac{d_1b_0}{d_0b_1+d_1b_0}\alpha w\quad\text{and}\quad p_1(\alpha)=p_1^*-\frac{d_0b_1}{d_0b_1+d_1b_0}\alpha w.$$

The changes in profit, consumer surplus, and social welfare are given by:

$$\begin{aligned}
\mathcal{R}(\alpha)-\mathcal{R}(0)&=-\frac{d_0d_1}{d_0b_1+d_1b_0}(\alpha w)^2\le 0,\\
\mathcal{S}(\alpha)-\mathcal{S}(0)&=\frac{d_0d_1}{2(d_0b_1+d_1b_0)}\left[(b_1-b_0)\alpha w+(\alpha w)^2\right]\ge 0,\\
\mathcal{W}(\alpha)-\mathcal{W}(0)&=\frac{d_0d_1}{2(d_0b_1+d_1b_0)}\left[(b_1-b_0)\alpha w-(\alpha w)^2\right]\ge 0.
\end{aligned}$$

If $\tilde{\alpha}_n<\alpha\le 1$, then

$$p_0(\alpha)=b_0\quad\text{and}\quad p_1(\alpha)=b_0+(1-\alpha)w.$$

Let $\tilde{p}_1=p_1^*-\frac{d_0b_1}{d_1b_0}\frac{b_0-c}{2}$. Then, we have:

$$\begin{aligned}
\mathcal{R}(\alpha)-\mathcal{R}(\tilde{\alpha}_n)&=\frac{-w^2(\alpha-\tilde{\alpha}_n)^2-(b_1+c-2\tilde{p}_1)w(\alpha-\tilde{\alpha}_n)}{b_1}<0,\\
\mathcal{S}(\alpha)-\mathcal{S}(\tilde{\alpha}_n)&=\frac{2(b_1-\tilde{p}_1)w(\alpha-\tilde{\alpha}_n)+w^2(\alpha-\tilde{\alpha}_n)^2}{2b_1}>0,\\
\mathcal{W}(\alpha)-\mathcal{W}(\tilde{\alpha}_n)&=\frac{(2\tilde{p}_1-2c)w(\alpha-\tilde{\alpha}_n)-w^2(\alpha-\tilde{\alpha}_n)^2}{2b_1}>0.
\end{aligned}$$

We next prove these statements below.

For case (1), since $N_i(p)=p/2$, the analysis from the proof of Proposition 2 holds before p_0 reaches b_0, that is, when $\alpha\le\frac{d_0b_1+d_1b_0}{d_1b_0}\frac{b_0-c}{b_1-b_0}$.

For $\alpha>\frac{d_0b_1+d_1b_0}{d_1b_0}\frac{b_0-c}{b_1-b_0}$, p_0 stays at b_0 and $N_0(b_0)=\frac{b_0}{2}$. The gap in no-purchase valuation is $(1-\alpha)[N_1(p_1^*)-N_0(p_0^*)]$, and hence $N_1(p_1)=\frac{b_0}{2}+(1-\alpha)[N_1(p_1^*)-N_0(p_0^*)]$, i.e., $\frac{p_1}{2}=\frac{b_0}{2}+(1-\alpha)\frac{b_1-b_0}{4}$. Rearranging terms leads to $p_1 = b_0 + (1-\alpha)(b_1 - b_0)/2$. Substituting p_1 into the profit and consumer surplus functions yields our desired result.

For case (2), one can observe that the no-purchase valuation fairness and profit are not impacted if $p_0 > b_0$ or if $p_0 = b_0$. Thus, without loss of generality case (2) is subsumed by case (1). For a similar reason, case (3) is subsumed by case (1). □

Appendix C: Proofs of Propositions 6 and 7

Proof of Proposition 6. We prove the results of each part separately. Without loss of generality, we assume that the parameters b_i are indexed in increasing order.

(a) Demand fairness. Given α, let $q_i = q_i(\alpha) = \bar{F}_i(p_i(\alpha))$ be the optimal normalized demand for group i. The profit of group i given q_i is equal to $d_i q_i(b_i - c - b_i q_i)$. Let $q_i^* = q_i(0) = (b_i - c)/2$ be the optimal unconstrained normalized demand. We define $I_{dec}(\alpha) = \{i | q_i(\alpha) < q_i^*\}$ and $I_{inc}(\alpha) = \{i | q_i(\alpha) > q_i^*\}$ as the sets of groups with demand that decrease and increase relative to the unconstrained optimal solution, respectively. For each specific α, we do not need to consider the groups whose prices remain unchanged, because these groups do not contribute to the difference in social welfare.

Consider the normalized demand for group $i \in I_{dec}(\alpha)$. We next show that all the groups in $I_{dec}(\alpha)$ should have the same demand. Indeed, if there exist $i, j \in I_{dec}(\alpha)$ such that $q_i(\alpha) > q_j(\alpha)$, one can increase q_j such that $q_j = q_i(\alpha)$. By doing so, the fairness constraints still hold, and we arrive at a demand that is closer to q_j^*, and hence corresponds to a higher profit. As a result, for all $i \in I_{dec}(\alpha)$, the demand level must be the same. Similarly, all the groups in $I_{inc}(\alpha)$ must have the same demand. Let q_{dec} and q_{inc} be the demand levels of decreasing and increasing groups, respectively. One can also see that w.l.o.g., $q_{dec} - q_{inc} = (1-\alpha)|q_{N-1}^* - q_0^*|$.

Let $q_{inc}(\alpha)$ and $q_{dec}(\alpha)$ be the demand levels for $I_{inc}(\alpha)$ and $I_{dec}(\alpha)$, respectively. We first show that $q_{inc}(\alpha)$ (resp. $q_{dec}(\alpha)$) increases (resp. decreases) monotonically with α. First, note that given q_{inc} and q_{dec}, we can construct a solution for all the N groups, by setting $q_i = \min(\max(q_{inc}, q_i^*), q_{dec})$. Let $h(q_{inc}, q_{dec}) = \sum_{i=0}^{N-1} R_i(\bar{F}_i^{-1}(\min(\max(q_{inc}, q_i^*), q_{dec})))$ be the profit with respect to q_{inc} and q_{dec}. One can easily verify that $h(q_{inc}, q_{dec})$ is concave in the region $0 \le q_{inc} \le q_{dec} \le 1$. The optimization problem (2) can then be written as

$$\begin{aligned}
\max_{q_{inc}, q_{dec}} \quad & h(q_{inc}, q_{dec}) \\
\text{s.t.} \quad & q_{dec} - q_{inc} \le (1-\alpha)|q_{N-1}^* - q_0^*|, \\
& q_{inc} \le q_{dec}, \\
& q_{inc}, q_{dec} \in [q_0^*, q_{N-1}^*].
\end{aligned}$$

The KKT condition is given by

$$\begin{aligned}
\begin{bmatrix} \frac{\partial h}{\partial q_{inc}} \\ \frac{\partial h}{\partial q_{dec}} \end{bmatrix} = \mu_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \mu_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad (19) \\
q_{dec} - q_{inc} \le (1-\alpha)|q_{N-1}^* - q_0^*|, \\
q_{inc} - q_{dec} \le 0, \\
\mu_1 \left(q_{dec} - q_{inc} - (1-\alpha)|q_{N-1}^* - q_0^*|\right) = 0 \\
\mu_2 (q_{inc} - q_{dec}) = 0 \\
q_{inc}, q_{dec} \in [q_0^*, q_{N-1}^*], \\
\mu_1, \mu_2 \ge 0.
\end{aligned}$$

Since $q_{dec} - q_{inc} = (1-\alpha)|q^*_{N-1} - q^*_0| > 0$, we have that $\mu_2 = 0$ due to complementary slackness. Note that $\frac{\partial h}{\partial q_{inc}}$ is non-positive and monotonically decreasing in the range $[q^*_0, q_{dec}]$, whereas $\frac{\partial h}{\partial q_{inc}}$ is non-negative and monotonically decreasing in the range $[q_{inc}, q^*_{N-1}]$. With these facts in hand, we see that as α increases in Eq. (19), q_{inc} and q_{dec} move towards one another. Since their difference is monotonically decreasing with α, we have $q_{inc}(\alpha)$ monotonically increases and $q_{dec}(\alpha)$ monotonically decreases.

Since we have shown that $q_{inc}(\alpha)$ and $q_{dec}(\alpha)$ are monotone and move towards one another, it follows that the functions are also continuous since $q_{dec}(\alpha) - q_{inc}(\alpha) = (1-\alpha)(q^*_{N-1} - q^*_0)$. Consequently, the corresponding social welfare $\mathcal{W}(\alpha)$ is also continuous in α. Since $q_{inc}(\alpha)$ and $q_{dec}(\alpha)$ are monotone, $I_{inc}(\alpha)$ and $I_{dec}(\alpha)$ are also monotone, that is, $I_{inc}(\alpha_1) \subset I_{inc}(\alpha_2)$ and $I_{dec}(\alpha_1) \subset I_{dec}(\alpha_2)$ for any $\alpha_1 < \alpha_2$. We can then split $\alpha \in [0,1]$ into at most N non-overlapping intervals, based on the value of $I_{inc}(\alpha)$ and $I_{dec}(\alpha)$. For the first interval, we have $I_{inc}(\alpha) = \{0\}$ and $I_{dec}(\alpha) = \{N-1\}$. As α increases, we either add group 1 to I_{inc} or group $N-2$ to I_{dec}, and so on. Since the social welfare curve is continuous, it is enough to show that for each interval such that $I_{inc}(\alpha)$ and $I_{dec}(\alpha)$ are fixed, the social welfare is monotonically decreasing. By the continuity of the social welfare function, this translates to the social welfare being monotonically decreasing.

Suppose that for $\alpha \in [\alpha_1, \alpha_2]$, $I_{inc}(\alpha)$ and $I_{dec}(\alpha)$ are fixed. Recall that the normalized demand for i in $I_{inc}(\alpha)$ (or $I_{dec}(\alpha)$) are the same. The profit maximization problem (2) can be re-written as

$$\begin{aligned} \max_{q_i} \quad & \sum_{i\in I_{dec}} d_i q_i (b_i - b_i q_i - c) + \sum_{i \in I_{inc}} d_i q_i (b_i - b_i q_i - c) \\ \text{s.t.} \quad & |q_i - q_j| \le (1-\alpha)(q^*_{N-1} - q^*_0), \ \forall i \in I_{dec}, \ j \in I_{inc}, \\ & q_i \in [0,1], \ \forall i \in I_{dec} \cup I_{inc}, \end{aligned} \tag{20}$$

where the groups for which $p_i(\alpha) = p^*_i$ are not considered because they do not impact the optimal solution. Based on the above analysis, problem (20) reduces to

$$\begin{aligned} \max_{q_{dec}, q_{inc}} \quad & q_{dec}\left(\sum_{i\in I_{dec}} d_i b_i - \left(\sum_{i\in I_{dec}} d_i b_i\right) q_{dec} - \left(\sum_{i\in I_{dec}} d_i\right) c\right) + q_{inc}\left(\sum_{i\in I_{inc}} d_i b_i - \left(\sum_{i\in I_{inc}} d_i b_i\right) q_{inc} - \left(\sum_{i\in I_{inc}} d_i\right) c\right) \\ \text{s.t.} \quad & |q_{dec} - q_{inc}| = (1-\alpha)(q^*_{N-1} - q^*_0), \\ & q_{dec}, q_{inc} \in [0,1]. \end{aligned}$$

Interestingly, this is exactly the problem for the setting with two groups: (i) group *dec* has population $\sum_{i\in I_{dec}} d_i$ and parameter $b_{dec} = \frac{\sum_{i\in I_{dec}} d_i b_i}{\sum_{i \in I_{dec}} d_i}$, and (ii) group *inc* has population $\sum_{i\in I_{inc}} d_i$ and parameter $b_{inc} = \frac{\sum_{i\in I_{inc}} d_i b_i}{\sum_{i\in I_{inc}} d_i}$. We further point out that the consumer surplus for multiple groups can also be represented by these two new aggregate groups (*dec* and *inc*). To see this, note that given a normalized demand q, the consumer surplus of this group is given by $b_i q^2/2$, which is linear in b_i. Thus, the total consumer surplus from I_{dec} (resp. I_{inc}) is $\sum_{i\in I_{dec}} d_i b_i q^2_{dec}/2 = (\sum_{i\in I_{dec}} d_i) b_{dec} q^2_{dec}/2$. As a result, the two-group problem has exactly the same profit and consumer surplus as the multi-group problem. Following Proposition 3, the social welfare for the two-group problem always decreases with α. Thus, on each piece $[\alpha_1, \alpha_2]$, the social welfare is monotonically decreasing, and the social welfare function is continuous on $[0,1]$, which implies that $\mathcal{W}(\alpha)$ is monotonically decreasing.

(b) Surplus fairness. Given α, recall that $p_i(\alpha)$ is the optimal solution for group i. We define $I_{dec} = \{i|p_i(\alpha) > p_i^*\}$ and $I_{inc} = \{i|p_i(\alpha) < p_i^*\}$ as the sets of groups with surplus that decrease and increase with α relative to the unconstrained optimal solution, respectively. As before, we do not need to consider any group i where $p_i(\alpha) = p_i^*$ and thus the surplus remains $S_i(p_i^*)$. These groups do not contribute to the change in social welfare, $\mathcal{W}(\alpha) - \mathcal{W}(0)$. As in part (a), all the groups in I_{dec} (I_{inc}) share the same level of surplus, and the difference between the two sets is $(1-\alpha)|S_{N-1}(p_{N-1}^*) - S_0(p_0^*)|$.

We consider two cases separately: $p_0(\alpha) > 0$ and $p_0(\alpha) = 0$. When $p_0(\alpha) > 0$, we note that for group i, if the surplus is s_i, then the demand is given by $\sqrt{2s_i/b_i}$, and the price is given by $b_i - \sqrt{2b_i s_i}$. As a result, the profit from group i is equal to $(b_i - \sqrt{2b_i s_i} - c)\sqrt{2s_i/b_i} = (\sqrt{2b_i} - c\sqrt{2/b_i})\sqrt{s_i} - 2s_i$. Given that all the groups in I_{dec} (I_{inc}) have the same level of surplus, we use s_{dec} (s_{inc}) to denote the surplus for all the groups in the set. Then, the profit-maximization problem (2) can be re-written as

$$\begin{aligned} \max_{s_{dec}, s_{inc}} \quad & \sum_{i \in I_{dec}} d_i \left[(\sqrt{2b_i} - c\sqrt{2/b_i})\sqrt{s_{dec}} - 2s_{dec} \right] + \sum_{i \in I_{inc}} d_i \left[(\sqrt{2b_i} - c\sqrt{2/b_i})\sqrt{s_{inc}} - 2s_{inc} \right] \\ \text{s.t. } & |s_{dec} - s_{inc}| = (1-\alpha)|S_{N-1}(p_{N-1}^*) - S_0(p_0^*)|, \end{aligned} \tag{21}$$

where we relax the non-negativity constraints on the price as we already assume that $p_i(\alpha) > 0$. Note that $\sqrt{2b} - c\sqrt{2/b}$ is a strictly increasing function with respect to b for $b > 0$ and it ranges from negative infinity to infinity. Thus, there exists a unique b_{dec} such that

$$\sqrt{2b_{dec}} - c\sqrt{2/b_{dec}} = \frac{\sum_{i \in I_{dec}} d_i(\sqrt{2b_i} - c\sqrt{2/b_i})}{\sum_{i \in I_{dec}} d_i},$$

and a unique b_{inc} such that

$$\sqrt{2b_{inc}} - c\sqrt{2/b_{inc}} = \frac{\sum_{i \in I_{inc}} d_i(\sqrt{2b_i} - c\sqrt{2/b_i})}{\sum_{i \in I_{inc}} d_i}.$$

Therefore, Eq. (21) can be rewritten as

$$\begin{aligned} \max_{s_{dec}, s_{inc}} \quad & \left(\sum_{i \in I_{dec}} d_i \right) \left[(\sqrt{2b_{dec}} - c\sqrt{2/b_{dec}})\sqrt{s_{dec}} - 2s_{dec} \right] + \left(\sum_{i \in I_{inc}} d_i \right) \left[(\sqrt{2b_{inc}} - c\sqrt{2/b_{inc}})\sqrt{s_{inc}} - 2s_{inc} \right], \\ \text{s.t. } & |s_{dec} - s_{inc}| = (1-\alpha)|S_{N-1}(p_{N-1}^*) - S_0(p_0^*)|, \end{aligned} \tag{22}$$

which is equivalent to a two-group problem as in (1). The consumer surplus of (22) is also the same as (21), both of which are $\sum_{i \in I_{inc}} d_i s_{inc} + \sum_{i \in I_{dec}} d_i s_{dec}$. We next note that $\sqrt{2b} - c\sqrt{2/b}$ is strictly concave in b, and thus by our definition of b_{dec} and b_{inc} we have $b_{dec} \le \bar{b} := \sum_{i \in I_{dec}} d_i b_i / \sum_{i \in I_{dec}} d_i$ and $b_{inc} \le \underline{b} := \sum_{i \in I_{inc}} d_i b_i / \sum_{i \in I_{inc}} d_i$, where $\bar{b}$ and $\underline{b}$ are the weighted averages of b_i in I_{dec} and I_{inc}, respectively.

Recall that the surplus of a group with parameters d and b is $d\frac{3(b-c)^2}{8b}$. Thus, we have

$$\begin{aligned} \text{Social Welfare from (22)} &< (\sum_{i \in I_{dec}} d_i) \frac{3(b_{dec} - c)^2}{8b_{dec}} + (\sum_{i \in I_{inc}} d_i) \frac{3(b_{inc} - c)^2}{8b_{inc}} \\ &\le (\sum_{i \in I_{dec}} d_i) \frac{3(\bar{b} - c)^2}{8\bar{b}} + (\sum_{i \in I_{inc}} d_i) \frac{3(\underline{b} - c)^2}{8\underline{b}} \\ &\le \sum_{i \in I_{dec}} d_i \frac{3(b_i - c)^2}{8b_i} + \sum_{i \in I_{inc}} d_i \frac{3(b_i - c)^2}{8b_i} \\ &= \sum_{i \in I_{dec}} S_i(p_i^*) + \sum_{i \in I_{inc}} S_i(p_i^*). \end{aligned}$$

The first inequality follows from Proposition 4, where we have shown that the social welfare under surplus fairness is lower than the unconstrained value in the two-group case, and the fact that (22) is equivalent to a two-group setting with as discussed above. The second inequality follows from the facts that $b_{dec} \leq \bar{b}$ and $b_{inc} \leq \underline{b}$. The third inequality follows from Jensen's inequality, and the final equality follows by definition. Since the social welfare of (21) and (22) are equivalent, then we conclude that when $\alpha > 0$, the social welfare is below the social welfare in the unconstrained case (i.e., when $\alpha = 0$).

When $p_0(\alpha) = 0$, we let $\tilde{\alpha}$ be the smallest α such that $p_0(\alpha) = 0$. For all $\alpha > \tilde{\alpha}$, the lowest surplus level is fixed as $b_0/2$ and cannot be improved. The only way to satisfy the constraints is to increase the prices for the groups whose surpluses are still too high. As a result, the social welfare monotonically decreases for $\alpha > \tilde{\alpha}$. Thus, we have $\mathcal{W}(\alpha) < \mathcal{W}(\tilde{\alpha}) < \mathcal{W}(0)$ for any $\alpha \geq \tilde{\alpha}$.

(c) No-purchase valuation fairness. Given α, let $p_i(\alpha)$ be the optimal solution for group i. We define $I_{dec}(\alpha) = \{i | p_i(\alpha) < p_i^*\}$ and $I_{inc}(\alpha) = \{i | p_i(\alpha) > p_i^*\}$ as the sets of groups with prices that decrease and increase relative to the unconstrained optimal solution, respectively. As in demand fairness, all the groups in $I_{dec}(\alpha)$ or $I_{inc}(\alpha)$ share the same level of no-purchase valuation. Note that using a price higher than b_i cannot improve the no-purchase valuation, and thus $p_i(\alpha)$ is at most b_i. In this case, the no-purchase valuation is simply equal to half of $p_i(\alpha)$ (for linear demand), i.e., $N_i(p_i(\alpha)) = \frac{p_i(\alpha)}{2}$. As a result, all the groups in $I_{dec}(\alpha)$ or $I_{inc}(\alpha)$ share the same price level, and the price difference between the two sets is $(1-\alpha)|p_{N-1}^* - p_0^*|$.

Let $p_{inc}(\alpha)$ and $p_{dec}(\alpha)$ be the prices for $I_{inc}(\alpha)$ and $I_{dec}(\alpha)$, respectively. We first show that $p_{inc}(\alpha)$ (resp. $p_{dec}(\alpha)$) increases (resp. decreases) monotonically with α. First, note that given p_{inc} and p_{dec}, we can construct a solution for all the N groups, by setting $p_i = \min(\max(p_{inc}, q_i^*), p_{dec})$. Let $g(p_{inc}, p_{dec}) = \sum_{i=0}^{N-1} R_i(\min(\max(p_{inc}, p_i^*), p_{dec}))$ be the profit with respect to p_{inc} and p_{dec}. One can easily verify that $g(p_{inc}, p_{dec})$ is concave in the range $p_{inc} \in [0, \min(p_{dec}, b_0)]$ and $p_{dec} \in [p_{inc}, b_{N-1}]$. Optimization problem (2) can then be written as

$$\begin{aligned}
\max_{p_{inc}, p_{dec}} \quad & g(p_{inc}, p_{dec}) \\
\text{s.t.} \quad & p_{dec} - p_{inc} \leq (1-\alpha)|p_{N-1}^* - p_0^*|, \\
& p_{inc} - p_{dec} \leq 0 \\
& p_{inc} \in [p_0^*, b_0], p_{dec} \in [p_0^*, p_{N-1}^*].
\end{aligned}$$

When p_{inc} does not reach the boundary b_0, the KKT condition is given by

$$\begin{aligned}
\begin{bmatrix} \frac{\partial g}{\partial p_{inc}} \\ \frac{\partial g}{\partial p_{dec}} \end{bmatrix} &= \mu_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \mu_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad (23) \\
p_{dec} - p_{inc} &\leq (1-\alpha)|p_{N-1}^* - p_0^*| \\
p_{inc} - p_{dec} &\leq 0 \\
\mu_1 \left(p_{dec} - p_{inc} - (1-\alpha)|p_{N-1}^* - p_0^*|\right) &= 0 \\
\mu_2 (p_{inc} - p_{dec}) &= 0 \\
\mu_1, \mu_2 &\geq 0.
\end{aligned}$$

Since the price difference between the two sets is $(1-\alpha)|p^*_{N-1}-p^*_0|$, by complementary slackness, we have and $\mu_2=0$. Note that $\frac{\partial g}{\partial p_{inc}}$ is non-positive and monotonically decreasing in the feasible region; similarly, $\frac{\partial g}{\partial p_{dec}}$ is non-negative and monotonically decreasing in the feasible region. Therefore, before p_{inc} reaches b_0, when we increase α to maintain Eq. (23), one has to move p_{inc} and p_{dec} in opposite directions. Since their difference is monotonically decreasing with α, then $p_{inc}(\alpha)$ monotonically increases and $p_{dec}(\alpha)$ monotonically decreases. When $p_{inc}(\alpha)$ reaches the boundary b_0, to satisfy the fairness constraints, one has to decrease $p_{dec}(\alpha)$ monotonically, while $p_{inc}(\alpha)$ remains at b_0.

We now know that $p_{inc}(\alpha)$ and $p_{dec}(\alpha)$ are monotone. Since their gap is $(1-\alpha)(p^*_{N-1}-p^*_0)$, they are both continuous. Consequently, the corresponding social welfare is also continuous. As α increases from 0, p_0 is increasing and p_{N-1} is decreasing. Let $\tilde{\alpha}$ be the smallest α such that $p_0=b_0$ (if it exists). Then, for any $\alpha>\tilde{\alpha}$, since the no-purchase valuation from group 0 cannot be improved anymore, the price of group 0 (as well as all the groups in I_{inc}) remains at b_0, and the only way to decrease the differences in no-purchase valuation is to decrease the price of the remaining groups whose offered price is greater than b_0. By doing so, the social welfare must increase. We next show that for $\alpha\le\tilde{\alpha}$, $\mathcal{W}(\alpha)$ also increases monotonically, hence concluding the proof.

For $\alpha\le\tilde{\alpha}$, since $p_{inc}(\alpha)$ and $p_{dec}(\alpha)$ are monotone, $I_{inc}(\alpha)$ and $I_{dec}(\alpha)$ are also monotone, i.e., $I_{inc}(\alpha_1)\subset I_{inc}(\alpha_2)$ and $I_{dec}(\alpha_1)\subset I_{dec}(\alpha_2)$ for any $\alpha_1<\alpha_2$. We can then split $[0,\tilde{\alpha}]$ into at most N non-overlapping intervals, based on the value of $I_{inc}(\alpha)$ and $I_{dec}(\alpha)$. For the first interval, we have $I_{inc}(\alpha)=\{1\}$ and $I_{dec}(\alpha)=\{N\}$. As α increases, we either add group 2 to I_{inc} or group $N-1$ to I_{dec}, and so on. Since the social welfare curve is continuous, it is enough to show that for each interval such that $I_{inc}(\alpha)$ and $I_{dec}(\alpha)$ are fixed, the social welfare is monotonically increasing.

Suppose that $\alpha\in[\alpha_1,\alpha_2]$ and that $I_{inc}(\alpha), I_{dec}(\alpha)$ are fixed. Then, the set of tight constraints is also fixed, and we know that the prices for i in $I_{inc}(\alpha)$ or $I_{dec}(\alpha)$ are the same. The profit maximization problem is thus equivalent to

$$\begin{aligned}\max_{p_{inc},p_{dec}}\ & \sum_{i\in I_{inc}(\alpha)} d_i(p_{inc}-c)(1-\frac{p_{inc}}{b_i})+\sum_{i\in I_{dec}(\alpha)} d_i(p_{dec}-c)(1-\frac{p_{dec}}{b_i}) \\ \text{s.t.}\ & p_{dec}-p_{inc}=(1-\alpha)(p^*_{N-1}-p^*_0),\end{aligned}\tag{24}$$

where the boundary constraints are hidden because we already assume that the prices do not hit the boundary. Rearranging the terms in (24) leads to

$$\begin{aligned}\max_{p_{inc},p_{dec}}\ & d_{inc}(p_{inc}-c)(1-\frac{p_{inc}}{b_{inc}})+d_{dec}(p_{dec}-c)(1-\frac{p_{dec}}{b_{dec}}) \\ \text{s.t.}\ & p_{dec}-p_{inc}=(1-\alpha)(p^*_{N-1}-p^*_0),\end{aligned}\tag{25}$$

where $d_{inc}=\sum_{i\in I_{inc}(\alpha)} d_i$, $d_{dec}=\sum_{i\in I_{dec}(\alpha)} d_i$, and b_{inc}, b_{dec} are defined by

$$\frac{1}{b_{inc}}=\sum_{i\in I_{inc}(\alpha)}\frac{d_i}{d_{inc}}\frac{1}{b_i},\quad \frac{1}{b_{dec}}=\sum_{i\in I_{dec}(\alpha)}\frac{d_i}{d_{dec}}\frac{1}{b_i}.\tag{26}$$

As a result, problem (24) is equivalent to a problem with two groups, *inc* and *dec*. Using Proposition 5, the social welfare with respect to the aggregate groups *inc* and *dec* is always increasing with α. We next show that the total social welfare of group $i \in I_{dec}(\alpha) \cup I_{inc}(\alpha)$ has a constant difference relative to the social welfare from the two aggregate groups. The total social welfare for all the groups in I_{inc} is given by

$$\begin{aligned}
&\sum_{i \in I_{inc}(\alpha)} d_i \left[(p_{inc} - c)(1 - \frac{p_{inc}}{b_i}) + \frac{1}{2}(b_i - p_{inc})(1 - \frac{p_{inc}}{b_i}) \right] \\
=& \sum_{i \in I_{inc}(\alpha)} \left[-\frac{1}{2}\frac{d_i}{b_i} p_{inc}^2 + \frac{d_i}{b_i} p + \frac{d_i b_i}{2} - d_i c \right] \\
=& -\frac{1}{2}\Big(\sum_{i \in I_{inc}(\alpha)} \frac{d_i}{b_i} \Big) p_{inc}^2 + \Big(\sum_{i \in I_{inc}(\alpha)} \frac{d_i}{b_i} \Big) p_{inc} + \sum_{i \in I_{inc}(\alpha)} \frac{d_i b_i}{2} - c \sum_{i \in I_{inc}(\alpha)} d_i \\
=& -\frac{1}{2}\frac{d_{inc}}{b_{inc}} p_{inc}^2 + \frac{d_{inc}}{b_{inc}} p_{inc} + \frac{d_{inc} b_{inc}}{2} - d_{inc} c + \left(\sum_{i \in I_{inc}(\alpha)} \frac{d_i b_i}{2} - \frac{d_{inc} b_{inc}}{2} \right),
\end{aligned} \tag{27}$$

where the first four terms in Eq. (27),$-\frac{1}{2}\frac{d_{inc}}{b_{inc}} p_{inc}^2 + \frac{d_{inc}}{b_{inc}} p + \frac{d_{inc} b_{inc}}{2} - d_{inc} c$, equal to the social welfare of the aggregate group *inc*. The same result also holds for group *dec*. Hence, the total welfare of all the groups in I_{inc} differs from the social welfare of the aggregate group *inc* by a constant term. By using Proposition 5, the social welfare of the aggregate groups *inc* and *dec* are monotonically increasing on $[\alpha_1, \alpha_2]$, and the social welfare of all the groups within $I_{inc} \cup I_{dec}$ increases monotonically. Since the social welfare is a continuous function with at most N pieces, and it increases with α on each piece, we conclude that $\mathcal{W}(\alpha)$ is increasing for $\alpha \in [0,1]$. □

Proof of Proposition 7. (a) Recall that we assume $b_0 < b_1 < \cdots < b_{N-1}$. In addition, the unconstrained optimal prices, $p_i^* = (b_i + c)/2$, are also in increasing order. One can verify that $p_0^* \le p_i(\alpha) \le p_{N-1}^*$. For $\alpha < 1 - \frac{\max\{p_{N-2}^* - p_0^*, p_{N-1}^* - p_1^*\}}{p_{N-1}^* - p_0^*}$, we have $(1-\alpha)(p_{N-1}^* - p_0^*) > \max\{p_{N-2}^* - p_0^*, p_{N-1}^* - p_1^*\}$, i.e., the required price range is large enough such that the prices for groups 2 to $N-1$ remain equal to p_i^*, and we only need to optimize the prices for groups 0 and N. As a result, the problem reduces to a two-group problem, and the desired result follows directly from Proposition 2.

(b) Recall from Table 2 that if a group i has positive demand, then the no-purchase valuation metric in the case of linear demand is $N_i(p_i(\alpha)) = \frac{p_i(\alpha)}{2}$. Thus, when all groups have positive demand, ensuring price fairness is equivalent to ensuring no-purchase valuation fairness. Consequently, our result follows immediately from Proposition 6.

(c) We provide a proof by example. On the right panel of Fig. 2, when $\alpha \in [0.47, 0.62]$, one can see that group 1 is excluded, but $\mathcal{W}(\alpha) > \mathcal{W}(0)$. On the other hand, when $\alpha > 0.62$, both groups 1 and 2 are excluded, and $\mathcal{W}(\alpha) < \mathcal{W}(0)$. □

Appendix D: Proof of Proposition 8

Proof. *(a) Price Fairness.* Let Δp_{xy} be the absolute value of the price change for group xy. Let the unconstrained weighted average price be $\bar{p}_i^* = \frac{d_{i0} p_{i0}^* + d_{i1} p_{i1}^*}{d_{i0} + d_{i1}}$, $i = 0, 1$, where $p_{xy}^* = \frac{b_{xy} + c}{2}$. Without loss of generality, we assume that $\bar{p}_1 > \bar{p}_0$. As α increases, p_{10} and p_{11} decrease, whereas p_{00} and p_{01} increase. The optimization problem is given by:

$$\begin{aligned}
\min\ & \frac{d_{00}}{b_{00}} \Delta p_{00}^2 + \frac{d_{01}}{b_{01}} \Delta p_{01}^2 + \frac{d_{10}}{b_{10}} \Delta p_{10}^2 + \frac{d_{11}}{b_{11}} \Delta p_{11}^2 \\
\text{s.t. } & \frac{d_{00}}{d_{00} + d_{01}} \Delta p_{00} + \frac{d_{01}}{d_{00} + d_{01}} \Delta p_{01} + \frac{d_{10}}{d_{10} + d_{11}} \Delta p_{10} + \frac{d_{11}}{d_{10} + d_{11}} \Delta p_{11} = \alpha(\bar{p}_1^* - \bar{p}_0^*) \\
& \Delta p_i \ge 0.
\end{aligned}$$

Here, we omit the upper-bound constraints as we only consider the case when $p_{xy} \in (0, b_{xy})$.

By solving the KKT conditions, we obtain:

$$\begin{aligned}
&\frac{1}{b_{x0}}\Delta p_{x0} = \frac{1}{b_{x1}}\Delta p_{x1},\\
&(d_{00}+d_{01})\frac{1}{b_{00}}\Delta p_{00} = (d_{10}+d_{11})\frac{1}{b_{10}}\Delta p_{10},\\
&\frac{d_{00}\Delta p_{00}+d_{01}\Delta p_{01}}{d_{00}+d_{01}} + \frac{d_{10}\Delta p_{10}+d_{11}\Delta p_{11}}{d_{10}+d_{11}} = \alpha(\bar{p}_1^* - \bar{p}_0^*).
\end{aligned}$$

Solving the above equations leads to

$$\Delta p_{00} = \frac{b_{00}\alpha w}{\frac{d_{00}b_{00}+d_{01}b_{01}}{d_{00}+d_{01}} + \frac{d_{00}+d_{01}}{(d_{10}+d_{11})^2}(d_{10}b_{10}+d_{11}b_{11})},\ \Delta p_{01} = \frac{b_{01}\alpha w}{\frac{d_{00}b_{00}+d_{01}b_{01}}{d_{00}+d_{01}} + \frac{d_{00}+d_{01}}{(d_{10}+d_{11})^2}(d_{10}b_{10}+d_{11}b_{11})},$$

$$\Delta p_{10} = \frac{b_{10}\alpha w}{\frac{d_{10}b_{10}+d_{11}b_{11}}{d_{10}+d_{11}} + \frac{d_{10}+d_{11}}{(d_{00}+d_{01})^2}(d_{00}b_{00}+d_{01}b_{01})},\ \Delta p_{11} = \frac{b_{11}\alpha w}{\frac{d_{10}b_{10}+d_{11}b_{11}}{d_{10}+d_{11}} + \frac{d_{10}+d_{11}}{(d_{00}+d_{01})^2}(d_{00}b_{00}+d_{01}b_{01})},$$

where $w = \bar{p}_1^* - \bar{p}_0^*$.

By substituting the above expressions into the profit and consumer surplus functions, we obtain:

$$\begin{aligned}
\mathcal{R}(\alpha) - \mathcal{R}(0) &= -\frac{(d_{00}+d_{01})^2(d_{10}+d_{11})^2}{(b_{00}d_{00}+b_{01}d_{01})(d_{10}+d_{11})^2 + (b_{10}d_{10}+b_{11}d_{11})(d_{00}+d_{01})^2}(\alpha w)^2,\\
\mathcal{S}(\alpha) - \mathcal{S}(0) &= \frac{(d_{00}+d_{01})^2(d_{10}+d_{11})^2}{2(b_{00}d_{00}+b_{01}d_{01})(d_{10}+d_{11})^2 + (b_{10}d_{10}+b_{11}d_{11})(d_{00}+d_{01})^2}\left[2(\bar{p}_1^* - \bar{p}_0^*)\alpha w + (\alpha w)^2\right],\\
\mathcal{W}(\alpha) - \mathcal{W}(0) &= \frac{(d_{00}+d_{01})^2(d_{10}+d_{11})^2}{2(b_{00}d_{00}+b_{01}d_{01})(d_{10}+d_{11})^2 + (b_{10}d_{10}+b_{11}d_{11})(d_{00}+d_{01})^2}\left[2(\bar{p}_1^* - \bar{p}_0^*)\alpha w - (\alpha w)^2\right].
\end{aligned}$$

Note that $\bar{p}_1 - \bar{p}_0 > 0$ (by assumption), so that before giving up a group, the social welfare is monotonically increasing for any $\alpha \in [0,1]$.

(b) Demand Fairness. For demand fairness, we assume that group 0 has a lower weighted average demand. Hence, p_{00} and p_{01} decrease, whereas p_{10} and p_{11} increase. The optimization problem is given by:

$$\begin{aligned}
\min\ & \frac{d_{00}}{b_{00}}\Delta p_{00}^2 + \frac{d_{01}}{b_{01}}\Delta p_{01}^2 + \frac{d_{10}}{b_{10}}\Delta p_{10}^2 + \frac{d_{11}}{b_{11}}\Delta p_{11}^2\\
\text{s.t. } & \frac{d_{00}}{d_{00}+d_{01}}\frac{\Delta p_{00}}{b_{00}} + \frac{d_{01}}{d_{00}+d_{01}}\frac{\Delta p_{01}}{b_{01}} + \frac{d_{10}}{d_{10}+d_{11}}\frac{\Delta p_{10}}{b_{10}} + \frac{d_{11}}{d_{10}+d_{11}}\frac{\Delta p_{11}}{b_{11}} = \alpha K\\
& \Delta p_i \geq 0,
\end{aligned}$$

where

$$\begin{aligned}
K &= \frac{d_{10}}{d_{10}+d_{11}}\frac{b_{10}-c}{2b_{10}} + \frac{d_{11}}{d_{10}+d_{11}}\frac{b_{11}-c}{2b_{11}} - \frac{d_{00}}{d_{00}+d_{01}}\frac{b_{00}-c}{2b_{00}} - \frac{d_{01}}{d_{00}+d_{01}}\frac{b_{01}-c}{2b_{01}}\\
&= \frac{c\left[b_{10}b_{11}(d_{10}+d_{11})(b_{01}d_{00}+b_{00}d_{01}) - b_{00}b_{01}(d_{00}+d_{01})(b_{10}d_{11}+b_{11}d_{10})\right]}{2b_{00}b_{01}b_{10}b_{11}(d_{00}+d_{01})(d_{10}+d_{11})} > 0
\end{aligned}$$

is the initial difference in weighted average demand. By solving the KKT conditions, we obtain:

$$\Delta p_{00} = \Delta p_{01} = \frac{d_{10}+d_{11}}{2(d_{00}+d_{01}+d_{10}+d_{11})}\alpha K,$$

$$\Delta p_{10} = \Delta p_{11} = \frac{d_{00}+d_{01}}{2(d_{00}+d_{01}+d_{10}+d_{11})}\alpha K.$$

We next consider the change in social welfare. The profit loss is

$$\frac{d_{00}}{b_{00}}\Delta p_{00}^2+\frac{d_{01}}{b_{01}}\Delta p_{01}^2+\frac{d_{10}}{b_{10}}\Delta p_{10}^2+\frac{d_{11}}{b_{11}}\Delta p_{11}^2,$$

the consumer surplus change is

$$d_{00}\frac{b_{00}-c}{2b_{00}}\Delta p_{00}+d_{01}\frac{b_{01}-c}{2b_{01}}\Delta p_{01}-d_{10}\frac{b_{10}-c}{2b_{10}}\Delta p_{10}-d_{11}\frac{b_{11}-c}{2b_{11}}\Delta p_{11}+\frac{d_{00}}{2b_{00}}\Delta p_{00}^2+\frac{d_{01}}{2b_{01}}\Delta p_{01}^2+\frac{d_{10}}{2b_{10}}\Delta p_{10}^2+\frac{d_{11}}{2b_{11}}\Delta p_{11}^2,$$

and the social welfare change is

$$d_{00}\frac{b_{00}-c}{2b_{00}}\Delta p_{00}+d_{01}\frac{b_{01}-c}{2b_{01}}\Delta p_{01}-d_{10}\frac{b_{10}-c}{2b_{10}}\Delta p_{10}-d_{11}\frac{b_{11}-c}{2b_{11}}\Delta p_{11}-\frac{d_{00}}{2b_{00}}\Delta p_{00}^2-\frac{d_{01}}{2b_{01}}\Delta p_{01}^2-\frac{d_{10}}{2b_{10}}\Delta p_{10}^2-\frac{d_{11}}{2b_{11}}\Delta p_{11}^2.$$

Note that the second-order term in the social welfare is always decreasing with α, so that we only need to focus on the linear terms. By substituting Δp_{xy}, we obtain:

$$\begin{aligned}&d_{00}\frac{b_{00}-c}{2b_{00}}\Delta p_{00}+d_{01}\frac{b_{01}-c}{2b_{01}}\Delta p_{01}-d_{10}\frac{b_{10}-c}{2b_{10}}\Delta p_{10}-d_{11}\frac{b_{11}-c}{2b_{11}}\Delta p_{11}\\=&\frac{-c\left[b_{10}b_{11}(d_{10}+d_{11})(b_{01}d_{00}+b_{00}d_{01})-b_{00}b_{01}(d_{00}+d_{01})(b_{10}d_{11}+b_{11}d_{10})\right]}{4b_{00}b_{01}b_{10}b_{11}(d_{00}+d_{01}+d_{10}+d_{11})}\alpha K.\end{aligned}\tag{28}$$

Note that the numerator in Eq. (28) equals to the numerator of $-K$, and thus is negative by assumption. Hence, Eq. (28) decreases with α. Together with the fact that $-\frac{d_{00}}{2b_{00}}\Delta p_{00}^2-\frac{d_{01}}{2b_{01}}\Delta p_{01}^2-\frac{d_{10}}{2b_{10}}\Delta p_{10}^2-\frac{d_{11}}{2b_{11}}\Delta p_{11}^2$ decreases with α, we conclude that the social welfare always decreases with α.

(c) Surplus Fairness. Finally, for surplus fairness, we follow the same idea as in Lemma 1. We assume that group 0 has a lower weighted average surplus. Hence, p_{00} and p_{01} decrease, whereas p_{10} and p_{11} increase. The optimization problem is given by:

$$\begin{aligned}\min\ &\frac{d_{00}}{b_{00}}\Delta p_{00}^2+\frac{d_{01}}{b_{01}}\Delta p_{01}^2+\frac{d_{10}}{b_{10}}\Delta p_{10}^2+\frac{d_{11}}{b_{11}}\Delta p_{11}^2\\ \text{s.t. }&\frac{d_{00}}{d_{00}+d_{01}}\Big(\frac{b_{00}-c}{2b_{00}}\Delta p_{00}+\frac{1}{2b_{00}}\Delta p_{00}^2\Big)+\frac{d_{01}}{d_{00}+d_{01}}\Big(\frac{b_{01}-c}{2b_{01}}\Delta p_{01}+\frac{1}{2b_{01}}\Delta p_{01}^2\Big)\\&+\frac{d_{10}}{d_{10}+d_{11}}\Big(\frac{b_{10}-c}{2b_{10}}\Delta p_{10}-\frac{1}{2b_{10}}\Delta p_{10}^2\Big)+\frac{d_{11}}{d_{10}+d_{11}}\Big(\frac{b_{11}-c}{2b_{11}}\Delta p_{11}-\frac{1}{2b_{11}}\Delta p_{11}^2\Big)=\alpha K\\&\Delta p_i\ge 0,\end{aligned}\tag{29}$$

where

$$K=\frac{d_{10}}{d_{10}+d_{11}}\frac{(b_{10}-c)^2}{8b_{10}}+\frac{d_{11}}{d_{10}+d_{11}}\frac{(b_{11}-c)^2}{8b_{11}}-\frac{d_{00}}{d_{00}+d_{01}}\frac{(b_{00}-c)^2}{8b_{00}}-\frac{d_{01}}{d_{00}+d_{01}}\frac{(b_{01}-c)^2}{8b_{01}}>0$$

corresponds to the initial difference. The KKT conditions are given by:

$$\begin{bmatrix}2\frac{d_{00}}{b_{00}}\Delta p_{00}\\2\frac{d_{01}}{b_{01}}\Delta p_{01}\\2\frac{d_{10}}{b_{10}}\Delta p_{10}\\2\frac{d_{11}}{b_{11}}\Delta p_{11}\end{bmatrix}=\mu\begin{bmatrix}\frac{d_{00}}{d_{00}+d_{01}}\left(\frac{b_{00}-c}{2b_{00}}-\frac{1}{b_{00}}\Delta p_{00}\right)\\\frac{d_{01}}{d_{00}+d_{01}}\left(\frac{b_{01}-c}{2b_{01}}-\frac{1}{b_{01}}\Delta p_{01}\right)\\\frac{d_{10}}{d_{10}+d_{11}}\left(\frac{b_{10}-c}{2b_{10}}-\frac{1}{b_{10}}\Delta p_{10}\right)\\\frac{d_{11}}{d_{10}+d_{11}}\left(\frac{b_{11}-c}{2b_{11}}-\frac{1}{b_{11}}\Delta p_{11}\right)\end{bmatrix}$$
$$Eq.\ (29),\Delta p_{xy}\ge 0,\mu\ge 0.$$

These conditions can reformulated as

$$\begin{aligned}2\frac{d_{00}}{b_{00}}\Delta p_{00}&=\frac{\frac{d_{00}}{d_{00}+d_{01}}\left(\frac{b_{00}-c}{2b_{00}}-\frac{1}{b_{00}}\Delta p_{00}\right)}{\frac{d_{01}}{d_{00}+d_{01}}\left(\frac{b_{01}-c}{2b_{01}}-\frac{1}{b_{01}}\Delta p_{01}\right)}2\frac{d_{01}}{b_{01}}\Delta p_{01}\\&=\frac{\frac{d_{00}}{d_{00}+d_{01}}\left(\frac{b_{00}-c}{2b_{00}}-\frac{1}{b_{00}}\Delta p_{00}\right)}{\frac{d_{10}}{d_{10}+d_{11}}\left(\frac{b_{10}-c}{2b_{10}}-\frac{1}{b_{10}}\Delta p_{10}\right)}2\frac{d_{10}}{b_{10}}\Delta p_{10}=\frac{\frac{d_{00}}{d_{00}+d_{01}}\left(\frac{b_{00}-c}{2b_{00}}-\frac{1}{b_{00}}\Delta p_{00}\right)}{\frac{d_{11}}{d_{10}+d_{11}}\left(\frac{b_{11}-c}{2b_{11}}-\frac{1}{b_{11}}\Delta p_{11}\right)}2\frac{d_{11}}{b_{11}}\Delta p_{11}\end{aligned}\tag{30}$$

$$Eq.\ (29),\ \Delta p_{xy} \geq 0.$$

Using the same argument as in Lemma 1, we divide Eq. (30) and Eq. (29) by α and take the limit as α goes to 0:

$$-\frac{d_{00}}{b_{00}}p'_{00}(0) = -\frac{\frac{d_{00}}{d_{00}+d_{01}}\frac{b_{00}-c}{2b_{00}}}{\frac{d_{01}}{d_{00}+d_{01}}\frac{b_{01}-c}{2b_{01}}}\frac{d_{01}}{b_{01}}p'_{01}(0) = \frac{\frac{d_{00}}{d_{00}+d_{01}}\frac{b_{00}-c}{2b_{00}}}{\frac{d_{10}}{d_{10}+d_{11}}\frac{b_{10}-c}{2b_{10}}}\frac{d_{10}}{b_{10}}p'_{10}(0) = \frac{\frac{d_{00}}{d_{00}+d_{01}}\frac{b_{00}-c}{2b_{00}}}{\frac{d_{11}}{d_{10}+d_{11}}\frac{b_{11}-c}{2b_{11}}}\frac{d_{11}}{b_{11}}p'_{11}(0),$$

$$-\frac{d_{00}}{d_{00}+d_{01}}\frac{b_{00}-c}{2b_{00}}p'_{00}(0) - \frac{d_{01}}{d_{00}+d_{01}}\frac{b_{01}-c}{2b_{01}}p'_{01}(0) + \frac{d_{10}}{d_{10}+d_{11}}\frac{b_{10}-c}{2b_{10}}p'_{10}(0) + \frac{d_{11}}{d_{10}+d_{11}}\frac{b_{11}-c}{2b_{11}}p'_{11}(0) = K.$$

Solving the above system of equations, we obtain:

$$p'_{00}(0) = -\frac{\frac{b_{00}-c}{2(d_{00}+d_{01})}K}{C},\quad p'_{01}(0) = -\frac{\frac{b_{01}-c}{2(d_{00}+d_{01})}K}{C},\quad p'_{10}(0) = \frac{\frac{b_{10}-c}{2(d_{10}+d_{11})}K}{C},\quad p'_{11}(0) = \frac{\frac{b_{11}-c}{2(d_{10}+d_{11})}K}{C},$$

where $C > 0$ is the normalization constant. The initial social welfare derivative, $\mathcal{W}(0)'$, becomes $-d_{00}\bar{F}_{00}(p^*_{00})p'_{00}(0) - d_{01}\bar{F}_{01}(p^*_{01})p'_{01}(0) - d_{10}\bar{F}_{10}(p^*_{10})p'_{10}(0) - d_{11}\bar{F}_{11}(p^*_{11})p'_{11}(0)$. By substituting $p'_{xy}(0)$, we obtain:

$$\begin{aligned}\mathcal{W}(0)' =& -d_{00}\bar{F}_{00}(p^*_{00})p'_{00}(0) - d_{01}\bar{F}_{01}(p^*_{01})p'_{01}(0) - d_{10}\bar{F}_{10}(p^*_{10})p'_{10}(0) - d_{11}\bar{F}_{11}(p^*_{11})p'_{11}(0)\\ =& \frac{K}{C}\left(\frac{d_{00}}{d_{00}+d_{01}}\frac{(b_{00}-c)^2}{4b_{00}} + \frac{d_{01}}{d_{00}+d_{01}}\frac{(b_{01}-c)^2}{4b_{01}} - \frac{d_{10}}{d_{10}+d_{11}}\frac{(b_{10}-c)^2}{4b_{10}} - \frac{d_{11}}{d_{10}+d_{11}}\frac{(b_{11}-c)^2}{4b_{11}}\right)\\ =& \frac{K}{C}(-2K) < 0.\end{aligned}$$

This shows that the social welfare decreases at $\alpha = 0$.

(d) No-Purchase Valuation Fairness. For no-purchase valuation fairness, since we only consider the case without reaching the boundary, the solutions from both price fairness and no-purchase valuation fairness are the same, just as in Proposition 2 and Proposition 5.

□

Appendix E: On the Computation Complexity of Pricing with Multiple Groups

We show in Lemma 2 that the optimal solution can be found efficiently by reducing the N-group pricing problem (2) to a one-dimensional optimization problem.

LEMMA 2. *Assume that the profit function $R_i(p)$ is unimodal. Then, the pricing problem can be reduced to an one-dimension optimization problem.*

Proof of Lemma 2. Given α, we start by analyzing the structure of the optimal solution. We then propose an efficient way to compute the optimal solution.

Price fairness. Let $p_{min} = \min_i p^*_i$ and $p_{max} = \max_i p^*_i$. Given α, all the prices should be within $[p_{min}, p_{max}]$. Otherwise, if there exists $p_i(\alpha) < p_{min}$ for example, then setting $p_i = p_{min}$ will not violate the fairness constraints, but will lead to a higher profit since the profit function is unimodal. We define $I_{dec} = \{i | p_i(\alpha) < p^*_i\}$ and $I_{inc} = \{i | p_i(\alpha) > p^*_i\}$ as the sets of groups with prices that decrease and increase relative to the unconstrained optimal solution, respectively. It is not hard to see that all the groups in I_{dec} should have the same price. Indeed, if there exist $i, j \in I_{dec}$ such that $p_i(\alpha) > p_j(\alpha)$, one can increase p_j such that $p_i = p_j$.

Such a change will not violate the fairness constraints but will lead to a higher profit due to the unimodality of the profit function. As a result, we can use p_{inc} and p_{dec} to denote the prices of the groups in I_{inc} and I_{dec}, respectively. In addition, the constraint should be tight, i.e., $p_{dec} - p_{inc} = (1-\alpha)(p_{max} - p_{min})$, as otherwise, we can decrease p_{inc} such that the fairness constraint is not violated but the profit for the groups in I_{inc} increases.

From the discussion above, the decision space reduces to a single decision variable, p_{inc}. Indeed, given the optimal value of p_{inc}, $p_{dec} = p_{inc} + (1-\alpha)(p_{max} - p_{min})$. For each group, if $p_i^* < p_{inc}$, then $p_i(\alpha) = p_{inc}$; else if $p_i^* > p_{dec}$, then $p_i(\alpha) = p_{dec}$; else $p_i(\alpha) = p_i^*$.

Demand fairness. We follow a similar argument as for price fairness, but now search on the demand space. Let $q_i^* = \bar{F}_i(p_i^*)$ be the demand at the unconstrained optimal solution. Let $q_{min} = \min_i q_i^*$ and $q_{max} = \max_i q_i^*$. Given α, all the demand values should be in $[q_{min}, q_{max}]$. Otherwise, if there exists $q_i(\alpha) < q_{min}$ for example, then setting $q_i = q_{min}$ would not violate the fairness constraints, but would lead to a higher profit due to the unimodality of $R_i(\cdot)$. We define $I_{dec} = \{i | q_i(\alpha) < q_i^*\}$ and $I_{inc} = \{i | q_i(\alpha) > q_i^*\}$ as the sets of groups with demands that decrease and increase relative to the unconstrained optimal solution, respectively. As before, it is not hard to see that all the groups in I_{dec} should have the same demand. Indeed, if there exist $i, j \in I_{dec}$ such that $q_i(\alpha) > q_j(\alpha)$, one can increase q_j such that $q_i = q_j$. Such a change would not violate the fairness constraints but will lead to a higher profit due to the unimodality of the profit function. As a result, we can use q_{inc} and q_{dec} to denote the prices of the groups in I_{inc} and I_{dec}, respectively. In addition, the constraint should be tight, i.e., $q_{dec} - q_{inc} = (1-\alpha)(q_{max} - q_{min})$, as otherwise, we can decrease q_{inc} such that the fairness constraint is not violated but the profit for the groups in q_{inc} increases.

From the discussion above, the decision space reduces to a single decision variable, q_{inc}. Indeed, given the optimal value of q_{inc}, $q_{dec} = q_{inc} + (1-\alpha)(q_{max} - q_{min})$. For each group, if $q_i^* < q_{inc}$, then $q_i(\alpha) = q_{inc}$; else if $q_i^* > q_{dec}$, then $q_i(\alpha) = q_{dec}$; else $q_i(\alpha) = q_i^*$. The corresponding prices can then be computed by inverting the demand function $\bar{F_i}(\cdot)$.

Surplus fairness and no-purchase valuation fairness. The argument and the way of computing the optimal solution are essentially the same as for demand fairness, except that the decision variable becomes the surplus and the no-purchase valuation, respectively. □

Appendix F: Tested Instances in Section 5 and Additional Figures

F.1. Instances and Figures for Two-Group Experiments

For two-group cases, we test the following instances:

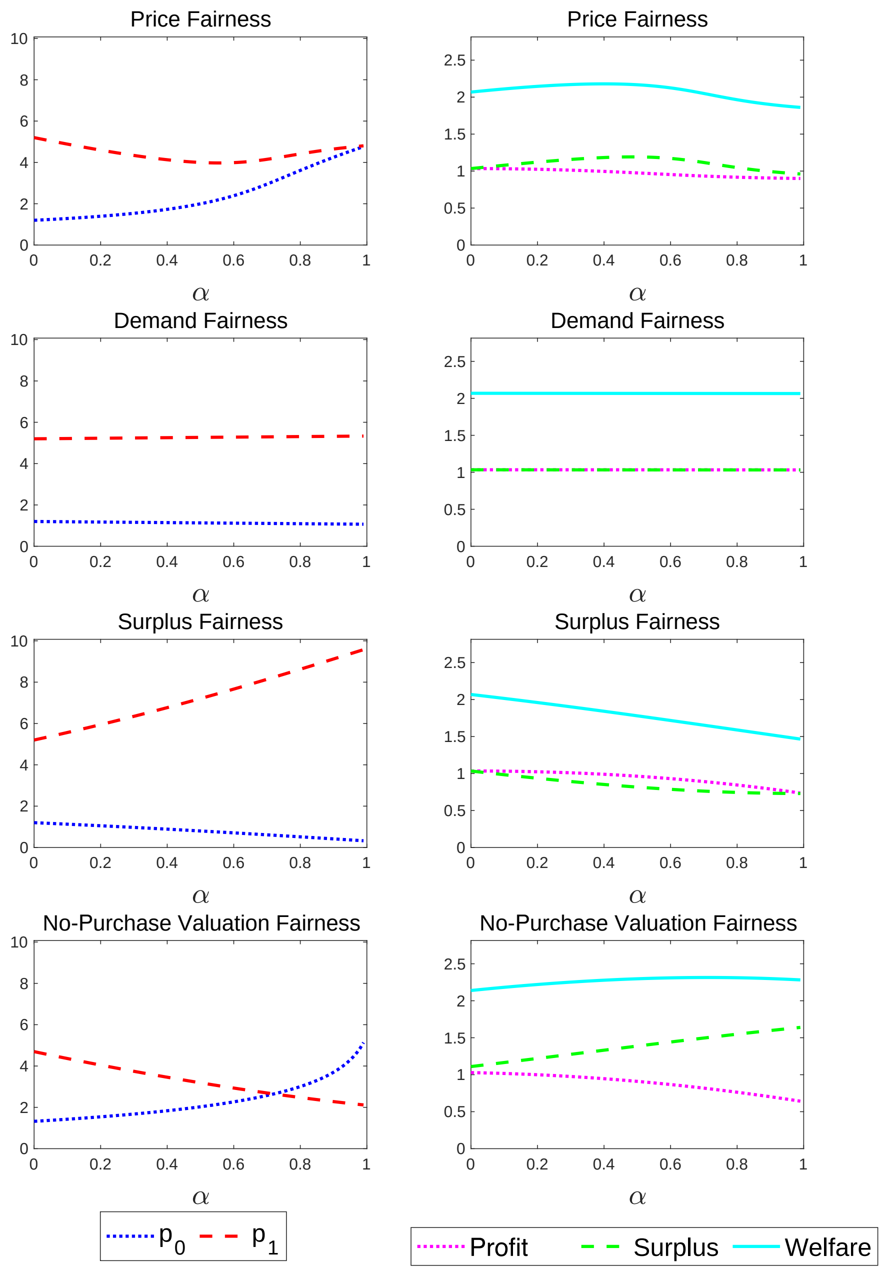
Exponential demand: For (d_0, d_1), we use (0.1, 0.9), (0.5, 0.5), and (0.9, 0.1). For (λ_0, λ_1), we use (1, 0.2) and (1, 2). For c, we use 0.1 and 2. We then test all the combinations.

Logistic demand: For (d_0, d_1), we use (0.1, 0.9), (0.5, 0.5), and (0.9, 0.1). For (k_0, k_1), we use (5,10), (10,5), and (5,5). For (λ_0, λ_1), we use (1, 0.2) and (1, 0.5). For c, we use 0.5 and 2. We then test all the combinations.

Log-log demand: For (d_0, d_1), we use (0.1, 0.9), (0.5, 0.5), and (0.9, 0.1). For (a_0, a_1), we use (2,1) and (1,2). For (β_0, β_1), we use (3, 1.8) and (3, 2.5). For c, we use 1 and 2. We then test all the combinations.

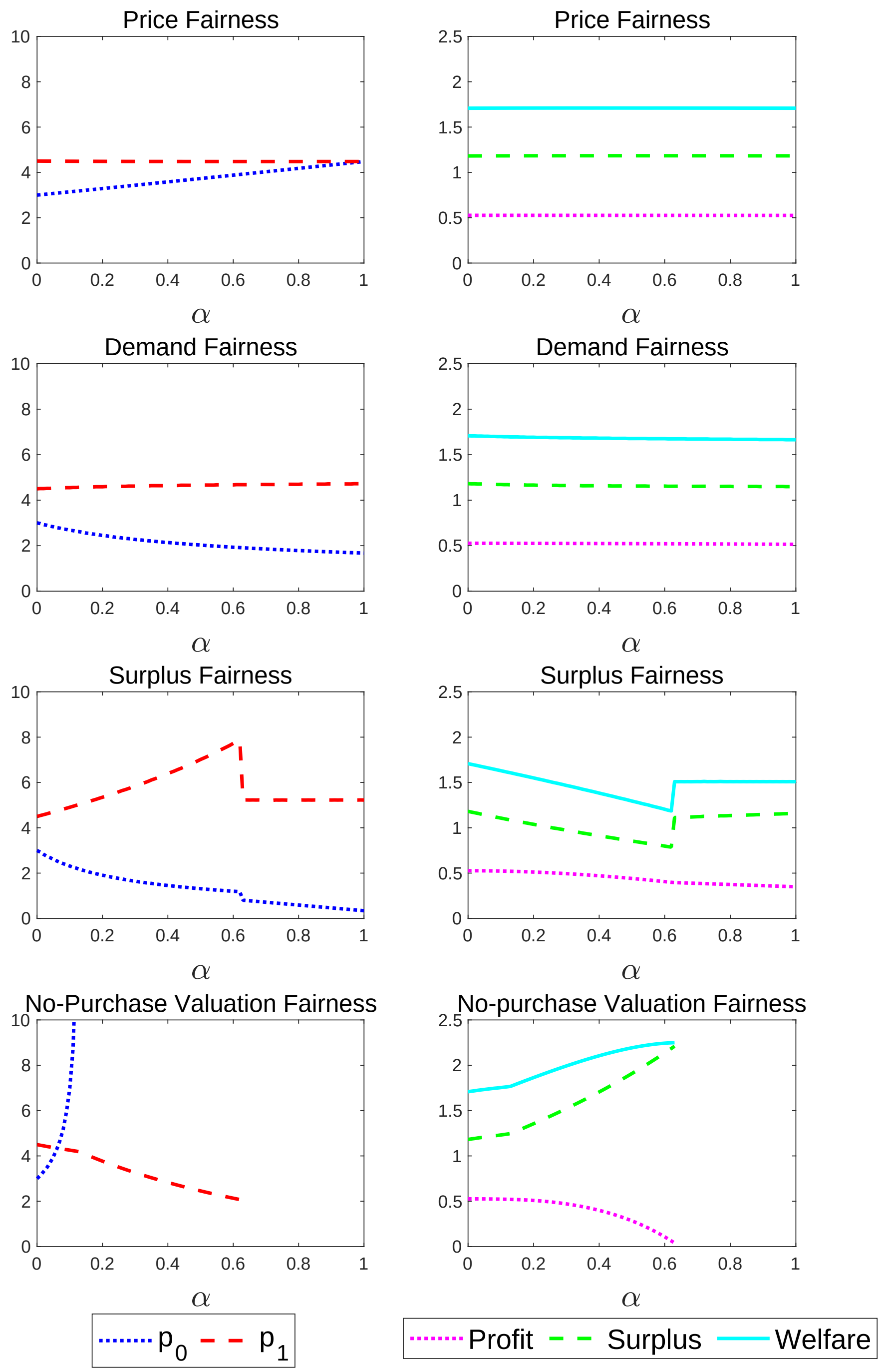
In Fig. 4, Fig. 3, and Fig. 5, we present the results for a representative example of each demand model.

Figure 4 **Impact of fairness under exponential demand (two groups).**



Note. Parameters: $d_0 = 0.5, d_1 = 0.5, \lambda_0 = 1, \lambda_1 = 0.2, c = 0.1$.

Figure 5 **Impact of fairness under log-log demand (two groups).**



Note. Parameters: $d_0 = 0.1, d_1 = 0.9, a_0 = 1, a_1 = 2, \beta_0 = 3, \beta_1 = 1.8, c = 2$. Note that the plot of no-purchase valuation fairness ends at $\alpha = 0.64$, since any larger α will result in an infeasible solution (because the demand of group 1 has reached 1 so that the no-purchase valuation is not well defined).

F.2. Instances and Figures for Five-Group Experiments

For five-group cases, we test the following instances:

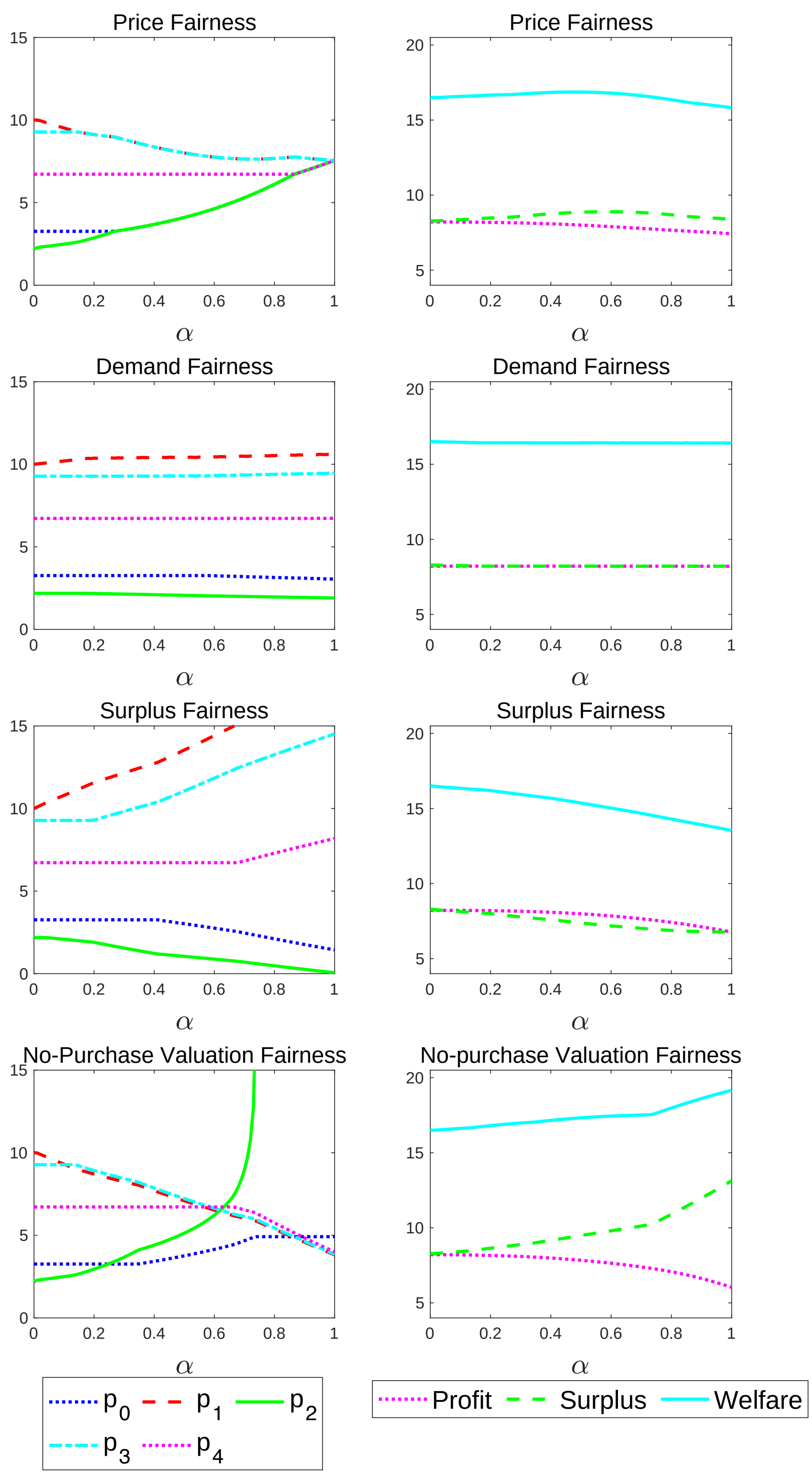
Exponential demand: We sample d_i uniformly between 0 and 1, and λ_i uniformly between 0.1 and 1. The value of c is set at 0.4

Logistic demand: We sample d_i uniformly between 0 and 1, λ_i uniformly between 0.1 and 1, and k_i uniformly between 3 and 10. The value of c is set at 2.

Log-log demand: We sample d_i uniformly between 0 and 1, β_i uniformly between 1.5 and 5. The value of c is set at 2. To make sure that $a_i(\beta_i - 1) < c\beta_i$, we sample a_i uniformly between $0.3c\beta_i/(\beta_i - 1)$ and $0.9c\beta_i/(\beta_i - 1)$.

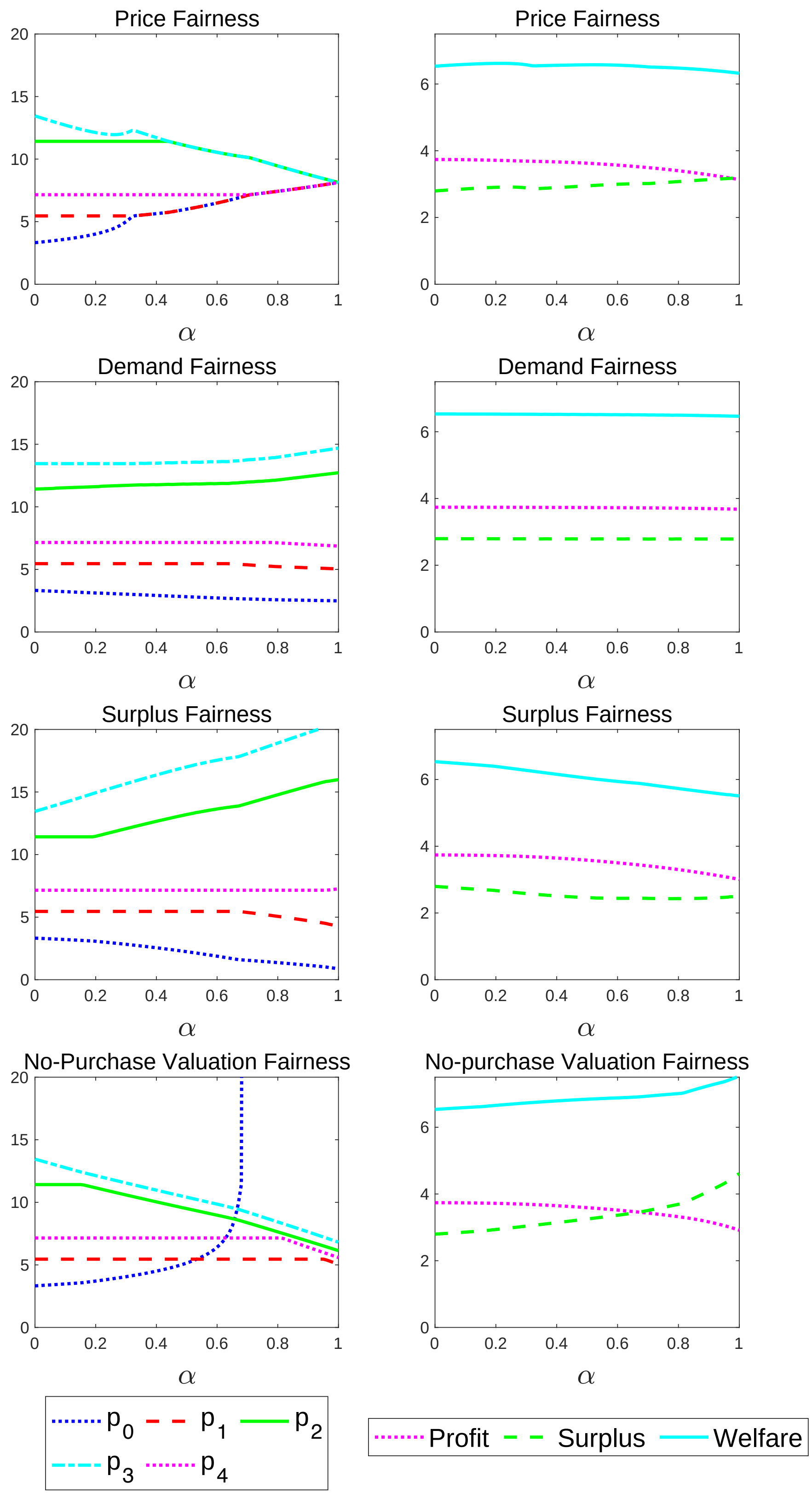
In Fig. 6, Fig. 7, and Fig. 8, we present the results for a representative example of each demand model.

Figure 6 **Impact of fairness under exponential demand (five groups).**



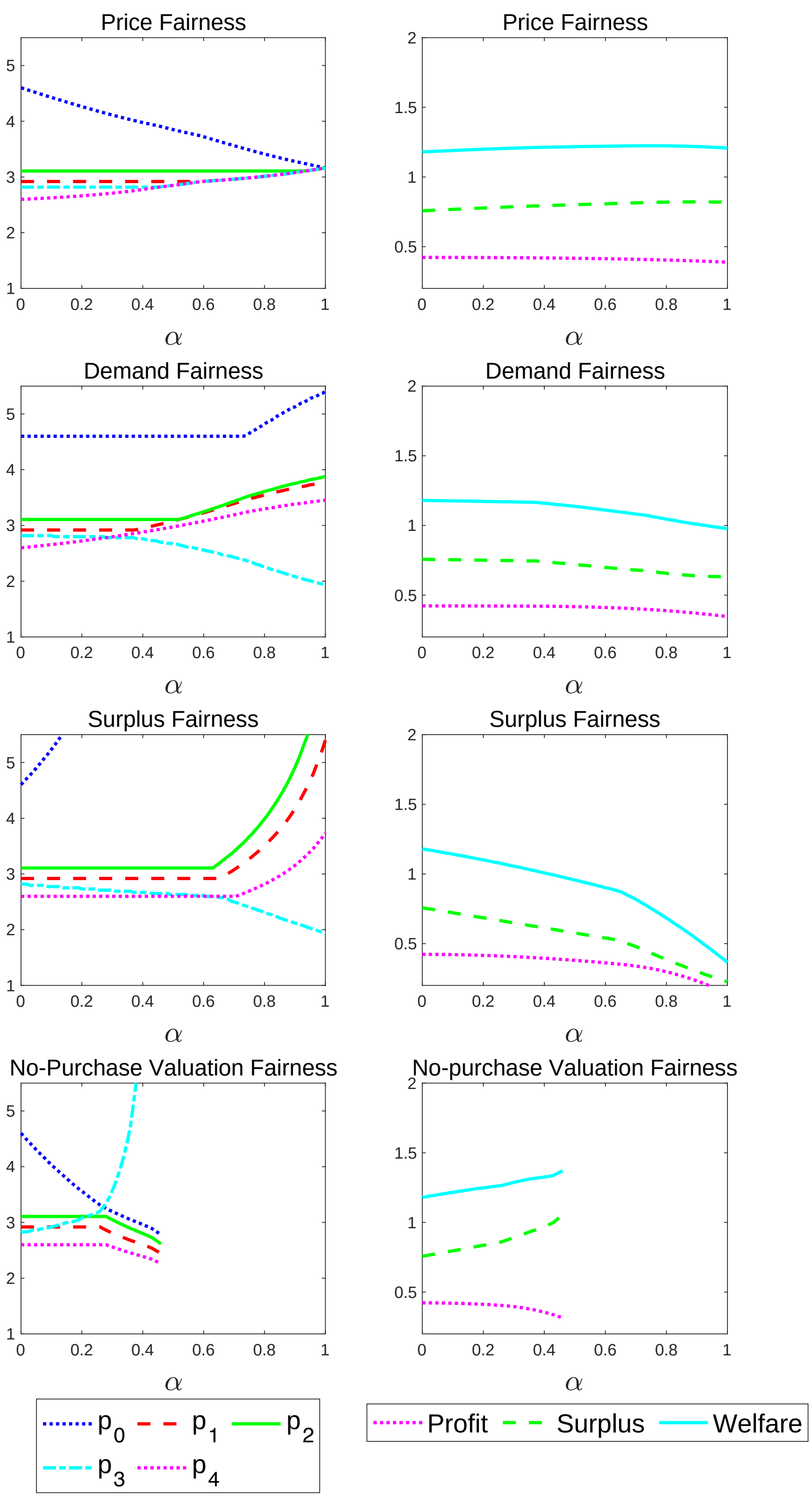
Note. Parameters: $\lambda = (0.35, 0.1, 0.56, 0.11, 0.16)$, $d = (0.98, 0.63, 0.49, 0.94, 0.87)$, $c = 0.4$.

Figure 7 **Impact of fairness under logistic demand (five groups).**



Note. Parameters: $\lambda = (0.99, 0.45, 0.2, 0.16, 0.32)$, $k = (8.28, 7.1, 9.48, 7.72, 6.32)$, $d = (0.23, 0.41, 0.17, 0.21, 0.63)$, $c = 2$.

Figure 8 Impact of fairness under log-log demand (five groups).



Note. Parameters: $\alpha = (1.89, 2.11, 2, 1.13, 2.25)$, $\beta = (1.77, 3.18, 2.81, 3.44, 4.34)$, $d = (0.32, 0.47, 0.09, 0.82, 0.12)$, $c = 2$. Note that the plot of no-purchase valuation fairness ends at $\alpha = 0.46$, since any larger α will result in an infeasible solution (because the demand of group 5 has reached 1 so that the no-purchase valuation is not well defined).