http://pubsonline.informs.org/journal/msom

MANUFACTURING & SERVICE OPERATIONS MANAGEMENT
Vol. 22, No. 2, March–April 2020, pp. 292–309
ISSN 1523-4614 (print), ISSN 1526-5498 (online)

Designing Price Incentives in a Network with Social Interactions

Maxime C. Cohen,[a] Pavithra Harsha[b]

[a] New York University Stern School of Business, New York, New York 10012; [b] IBM Research Thomas J. Watson Research Center, Yorktown Heights, New York 10598

Contact: maxime.cohen@stern.nyu.edu, http://orcid.org/0000-0002-2474-3875 (MCC); pharsha@us.ibm.com, http://orcid.org/0000-0002-6049-7739 (PH)

Received: February 1, 2017
Revised: March 10, 2018; May 14, 2018
Accepted: May 22, 2018
Published Online in Articles in Advance: August 26, 2019

https://doi.org/10.1287/msom.2018.0738

Abstract. ***Problem definition***: The recent ubiquity of social networks allows firms to collect vast amount of data on their customers and on their social interactions. We consider a setting in which a monopolist sells an indivisible good to consumers who are embedded in a social network. ***Academic/practical relevance***: This is an important problem as sellers can use available data to design and send targeted promotions that account for social externality effects and ultimately increase their profits. ***Methodology***: We capture the interactions among consumers using a broad class of nonlinear utility models. This class extends the existing models by explicitly capturing externalities from subsets of agents (communities or groups) and includes several existing models as special cases (e.g., full information version of the triggering model). Assuming complete information about the interactions, we model the optimal pricing problem as a two-stage game. First, the firm designs prices to maximize profits and then consumers choose whether to purchase the item. ***Results***: Under positive network externalities, we show the existence of a pure Nash equilibrium that is preferred by both the seller and the buyers. Using duality theory and integer-programming techniques, we reformulate the problem into a linear mixed-integer program (MIP). We derive efficient ways to optimally solve the MIP using its linear-programming relaxation for two pricing strategies: discriminative and uniform. Finally, we propose two intuitive heuristic algorithms to solve the problem for which we derive worst-case parametric performance bounds. ***Managerial implications***: We draw interesting insights on the structure of the optimal prices and the seller's profit. In particular, we quantify the effect on prices when using a nonlinear utility model relative to a linear model and identify settings with which it is beneficial to offer a price below cost to influential agents. Finally, we extend our model and results to the case in which the seller offers incentives (in addition to prices) to solicit actions so as to ensure network externality effects.

Supplemental Material: The online appendix is available at https://doi.org/10.1287/msom.2018.0738.

Keywords: pricing • influence models • social networks • integer programming • network externalities

1. Introduction

The recent ubiquity of social networks has revolutionized the way people interact and influence each other. The overwhelming success of social-networking platforms, such as Facebook and Twitter, allows firms to collect unprecedented volumes of data on their customers, their buying behavior, and their social interactions. The challenge faced by every firm is how to process this data and turn it into actionable policies to improve their competitive advantage. In this paper, we focus on designing effective pricing strategies to enhance the profits of a firm that sells indivisible goods (or services) to agents embedded in a social network.

Word-of-mouth communication between agents has always been an effective marketing tool. In recent times, word-of-mouth communication is just as likely to arise from social networks as from a neighbor across the fence. Consultants at The Conversation Group report that 65% of consumers who receive a recommendation from a contact on their social media have purchased the recommended product. In particular, personalized referrals from friends and family have been more effective in encouraging such purchases. Further, nearly 93% of social media users have either made or received a recommendation for a product or service. Academic research on consumer behavior shows that consumers' purchasing decisions are influenced by their reference groups (see, e.g., Iyengar et al. 2011). The previous examples clearly indicate that people influence their connections. They not only guide their purchasing behavior, but more importantly, alter their willingness to pay for various items. For example, when an individual buys a product and posts a positive review on the individual's Facebook or Yelp page, the individual does influence the individual's peers to purchase the same item and increases its valuation. The valuation increase can sometimes be nonlinear. For example, if many other people have already recommended this item, the marginal effect of a new recommender can have a relatively small increase. Alternatively, sometimes having a single friend who buys and recommends the item may be sufficient to trigger a significant increase in valuation.

An important feature of the products or services we consider in this paper is the *local (nonlinear) positive externalities*. This means that people *positively* influence each other's willingness to pay for an item (the setting with negative externalities is briefly discussed in Section 6.2). In addition, the item becomes more valuable to a person if many of the person's friends buy it even though there can be a decreasing (or increasing) marginal effect (submodular or supermodular). Examples of products with such effects include smartphones, tablets, fashion items, and cell phone plan subscriptions. Such positive externalities may be even more significant when a new generation of products is introduced to the market and people use social networks as a way to accelerate their friends' awareness about the item.

It is common practice that a very small number of highly influential people (e.g., certified bloggers) receive the item nearly for free to increase the awareness of the remaining population. Mark W. Schaefer (2012), the author of *Return on Influence*, reports, "For the first time, companies large and small can find these passionate influencers (using social networks), connect to them, and turn them into brand advocates." Therefore, it can be valuable for firms to identify these influential agents. As an example, many online sellers let consumers sign in with their Facebook account. Consequently, they have access to their personal information such as age, gender, geographical location, and number of friends as well as, more importantly, their network. Various sellers even build a Facebook page to advertise their firm through social platforms. For example, the large U.S. corporation Macy's has more than 14.68 million fans who liked its Facebook page (February 2018). These fans can claim offers via the social platform and, thus, directly influence their friends about purchasing. This interaction between the seller and the fan club allows the seller to keep its fan club engaged and to identify influencers. Ultimately, the seller can offer personalized prices or incentives to these influencers to increase the overall profitability.

In this paper, our goal is to develop a model that incorporates local nonlinear externalities among potential buyers and design efficient algorithms to compute the optimal prices that maximize the seller's profit. We formulate the optimal pricing problem as a two-stage game between the seller and the agents in the network in which the seller first offers prices and the agents then choose whether to purchase the (indivisible) item. The main contributions of the paper are as follows:

(1) Nonlinear additive utility models with externalities: We introduce and study a class of additive utility models. This class extends existing models by explicitly capturing externalities from subsets of buyers (communities or groups) and allowing a threshold on the number of agents needed to trigger the externality effect. Several commonly used models in the literature are included as special cases under full information (e.g., independent cascade model, linear threshold, and triggering model). In particular, the total value earned by an agent when purchasing the item is the sum of the agent's own valuation and the valuation derived from externalities of all subsets of friends (see Section 2.1). This is a broad class of nonlinear utility functions that can capture different influence structures, including special cases of supermodular and submodular, with respect to the number of neighbors.

(2) Reformulating the optimal pricing problem into an operational mixed-integer program (MIP). The strategic-complements nature of the second-stage game guarantees the existence of a pure strategy Nash equilibrium (under nonnegative externalities). Using duality theory, we derive equilibrium constraints and reformulate the two-stage problem into a nonconvex-integer program. We then transform it into an equivalent MIP using integer programming reformulation techniques. This resulting MIP holds under general externalities (positive or negative) and can be viewed as an operational pricing tool with which one can easily incorporate business rules on prices.

(3) Efficient optimal algorithms for discriminative and uniform pricing strategies. We develop efficient and scalable methods to optimally solve the MIP for two pricing strategies using the linear programming (LP) relaxation. We consider discriminative and uniform pricing strategies and present a solution method that is efficient (polynomial in the number of agents) and scalable to large networks. We also propose two heuristic algorithms (the *greedy expansion* and the *greedy removal*) that are intuitive and easy to implement, and we derive parametric bounds on their worst-case performance under nonnegative externalities.

(4) Insights on the structure of the optimal pricing strategy. Under discriminative prices, we show that the price of a buyer can be expressed as the sum of the buyer's own value and a markup term corresponding to the network externalities of agents who buy the item. Therefore, prices for influenced agents are higher (as expected). The seller's profit from network externalities comes from two types of agents: (1) high-valued customers who influence their neighbors and (2) low-valued customers who are highly influential and can sometimes be offered a price below cost. In addition, when comparing linear to nonlinear utility models, we show that, as we move from a linear model (with only pairwise interactions) to a nonlinear one (that includes

subsets of higher sizes), additional agents will buy, buyers will pay a higher price, and the profit increases. We also convey that a larger threshold on the minimum number of agents results in a smaller number of buyers and decreases the seller's profit.

(5) Price incentives that guarantee influence. We extend our model and results to optimally design both prices and incentives to solicit actions so as to ensure externality effects. This new model we introduce is more realistic and allows the seller to ensure that network externalities among agents occur by offering both a price and a discount (incentive) to each buyer. The buyer can then decide between (1) not buying the item, (2) buying the item at full price, and (3) buying the item while claiming the discount in exchange for influence actions specified by the seller (e.g., liking the product on social media or writing a review). Interestingly, the methods and results we develop extend to this richer model.

1.1. Literature Review

Models that incorporate local network externalities find their origins in papers by Farrell and Saloner (1985) and Katz and Shapiro (1985). These early papers assume that consumers are affected by the global consumption of all other consumers. In other words, the network effects are of a global nature; that is, the utility of a consumer depends directly on the behavior of the entire set of agents in the network. In our model, consumers only interact with a subset of agents, also known as their neighbors (or friends). Although interactions are of a local nature, the utility of each player may still depend on the global structure of the network given that each agent potentially interacts indirectly with a much larger set of agents. Models of local network externalities that explicitly account for the network structure have been proposed in various papers (e.g., Ballester et al. 2006 and Banerji and Dutta 2009). Several recent papers explicitly model the interactions among agents in social networks to study the network effects on marketing campaigns. The first among these are the papers on influence maximization (i.e., selecting the optimal set of nodes in a social network to maximize the spread) by Domingos and Richardson (2001) and Kempe et al. (2003), which aimed to identify influential agents in a network. Hartline et al. (2008), Akhlaghpour et al. (2010), and Arthur et al. (2009) extend this line of work to study optimal pricing strategies in networks. Hartline et al. (2008) focus on viral marketing strategies for revenue maximization in which agents are offered the product in a sequential manner and show that simple two-price strategies perform well relative to the optimal strategy, which is NP-hard. Akhlaghpour et al. (2010) extend this approach to a multistage model in which the seller sets different prices at each stage. Arthur et al. (2009) allow agents to buy the product with a certain probability if the product is recommended by their friends who purchased the item. These papers consider sequential purchases in which myopic consumers base their consumption decisions on the number of consumers who have already bought the product. In our paper, however, we consider a simultaneous purchasing decision for all agents in the network who are fully rational. Since the seminal work of Kempe et al. (2003) on the influence-maximization problem, several papers followed up on this topic. In Chen et al. (2010), the authors propose a new heuristic algorithm that is easily scalable to millions of nodes. The work in Borodin et al. (2010) extends the influence maximization problem to a competitive setting. The authors show that the problem becomes more challenging and that greedy approaches cannot be used anymore. Another recent paper is the work by Gunnec and Raghavan (2017) that investigates social network influence in the context of product design (share-of-choice problem). Finally, Nakkas and Xu (2015) study bargaining in two-sided supply chain networks and examine how valuation heterogeneity among manufacturers influences the equilibrium prices and the trading pattern of the supply chain network.

Our paper is also related to several studies in the marketing literature, especially in the field of social marketing (see, e.g., Andreasen 1994). Since the introduction of the Bass model, the diffusion of innovation has been an active area of research (see, e.g., Mahajan et al. 1991 and the references therein). This model assumes that potential adopters of an innovative product are influenced by mass media (called "innovators") and word of mouth (called "imitators"). A large number of marketing papers conduct empirical studies on the impact of word of mouth in the context of social networks (see, e.g., Goldenberg et al. 2001). Another stream of papers (both in economics and marketing) study models with *indirect network effects* (INE), which postulate that the utility of the primary product increases as more complements become available. Stremersch et al. (2007) provide a good marketing literature review on this topic. In Basu et al. (2003), the authors consider a model in which the utility of a product increases with the greater availability of complementary products and show that the INE can vary by product attributes. More recently, Lovett et al. (2013) present an empirical analysis on the relationship between brands and word of mouth. The authors argue that consumers spread the word on brands as a result of three drivers: social, emotional, and functional. Finally, several marketing and information systems papers consider the problem of running field experiments to identify causal estimates of social influence in networks or to asses the effectiveness of viral features in marketing campaigns (see, e.g., Aral and Walker 2011).

We model the pricing problem with simultaneous purchasing decisions as a two-stage game, in which the seller first sets the prices and then agents make their purchasing decisions. Rational behavior is captured by the Nash equilibrium. Three papers in this context are closely related to our work. Candogan et al. (2012) study optimal pricing strategies for a divisible good with linear utility functions under complete network information. The authors provide efficient algorithms to compute discriminative prices and the uniform optimal price and to show that the problem is NP-hard when the monopolist is restricted to two prespecified prices. Bloch and Querou (2013) and Chen et al. (2011) study the optimal pricing problem of an indivisible good with linear utility functions under incomplete information. Our work is in a similar light as the three aforementioned papers. In particular, we study the optimal pricing problem for indivisible items under a general class of nonlinear network externality models. The class of models we propose can be seen as a deterministic generalization (because of the perfect information assumption) of previous utility models (e.g., the *independent cascade* in Kempe et al. 2003 and the *triggering* model in Seeman and Singer 2013) and can capture submodular and supermodular functions. Our models allow explicitly capturing externalities of subsets of agents (communities or groups) and a threshold on the number of agents needed to trigger the externality effect. In addition, the techniques required to address the pricing problem under general utility models differ significantly from earlier papers. In particular, one cannot derive a closed-form solution for the optimal pricing problem as in Candogan et al. (2012) and Bloch and Querou (2013). Instead, the equilibrium in our setting can be characterized by a system of nonconvex constraints with integer variables. We use techniques from integer programming (IP) to reformulate the optimal pricing problem with the equilibrium constraints into a linear MIP. We refer the reader to the books by Nemhauser and Wolsey (1988) and Bertsimas and Weismantel (2005) for the IP techniques used in this paper.

1.2. Structure of the Paper

In Section 2, we describe our model, assumptions, and dynamics of the two-stage game. In Section 3, we show the existence of a pure strategy Nash equilibrium for the purchasing game (under nonnegative externalities). We use duality theory to formulate the problem as an MIP in Section 4 and derive efficient algorithms to optimally solve it for discriminative and uniform pricing strategies in Section 5. In Section 6, we propose two heuristic algorithms and discuss the setting with general externalities (positive or negative). Section 7 extends our results to the case in which the seller designs both prices and incentives to guarantee network externality effects. In Section 8, we present computational experiments. Finally, our conclusions are reported in Section 9. The proofs of the technical results are relegated to the online appendix.

2. Problem Setting

2.1. Utility Model with Network Externalities

Consider a monopolist selling an indivisible product to N agents, denoted by the set $\mathcal{I} = \{1, \ldots, N\}$, embedded in a social network. We denote the set of *value interactions* of agent $i \in \mathcal{I}$ by $G_i = \{g_{S,i} \mid S \subset \mathcal{I}\backslash\{i\}\}$, where the element $g_{S,i}$ represents the marginal increase in value that agent i obtains by owning the product when agents in S influence the agent.[1] This excludes the network effects resulting from subsets of S, each of which is modeled explicitly. Concrete examples are presented in the sequel. Note that the network considered in this paper can be represented as a weighted bipartite graph, $G(\mathcal{I}, \Delta_{\mathcal{I}}, E)$, where $\mathcal{I}$ denotes the set of agents, $\Delta_{\mathcal{I}}$ corresponds to the set of subsets of $\mathcal{I}$, and E represents the links between the groups so that the weight of a link determines the externality effect from one subset to an agent. We next define the threshold $\Gamma \geq 1$. For any set S, a minimum number of $\min\{\Gamma, |S|\}$ buyers in S are required to influence agent i. Here $|.|$ refers to the cardinality of a set. The term $g_{\emptyset,i}$ (also denoted by g_i) is the marginal value that agent i derives by owning the product. If agent j does not influence agent i, then all terms $g_{S,i}$ where $j \in S$ are zero. On the other hand, if agent j influences agent i, then at least one of the terms $g_{S,i}$ where $j \in S$ is nonzero. In this case, we refer to j as a *neighbor* (or friend) of i.

Assumption 1. *We make the following assumptions regarding the elements of G_i $\forall i \in \mathcal{I}$ and the corresponding utility model.*

(1) *The firm and the agents have perfect knowledge on externalities; that is, everyone knows G_i.*

(2) *Positive externality: The network externality of any set S on any agent i is nonnegative. That is, $g_{S,i} \geq 0$ for any $S \subset \mathcal{I}\backslash\{i\}$ and $i \in \mathcal{I}$.*

(3) *Additive model: The total value earned by an agent when purchasing the item is the sum of the agent's own valuation and the valuation derived from the network externalities of all subsets of friends who own the item. Mathematically, agent i's valuation is given by*

$$v_i(\alpha_i, \boldsymbol{\alpha}_{-i}) = \alpha_i \Big[\sum_{S \subset \mathcal{I}\backslash\{i\}} g_{S,i} \alpha_S \Big], \tag{1}$$

where $\alpha_i \in \{0, 1\}$ is a binary variable that represents the purchasing decision of agent i, $\boldsymbol{\alpha}_{-i}$ represents the vector of purchasing decisions of all agents but i, and α_S is a binary function that maps the purchasing decisions of agents in S to *a binary indicator which is one if agents in S influence i and zero otherwise. More specifically, we have*

$$\alpha_S = \max_{\substack{S' \subset S \\ |S'| = \min\{\Gamma, |S|\}}} \Big\{ \prod_{j \in S'} \alpha_j \Big\}. \tag{2}$$

Equation (2) captures the fact that agents in S influence i if at least Γ of them purchase the product. Note that we consider a setting with perfect information to make the analysis tractable and to draw some insights on our problem. This is a common assumption in several previous papers, such as Candogan et al. (2012). The positive externality assumption is also a commonly used condition that allows ensuring the existence of an equilibrium as we show in Theorem 1. The setting in which externalities can be negative is discussed in Section 6.2.

The additive model in Equation (1) together with Equation (2) capture a broad class of linear and nonlinear utility models with network externalities. In particular, it can be seen as a deterministic generalization (because of the perfect information assumption) of the independent cascade and the linear threshold models formulated in Kempe et al. (2003) and the triggering model studied in Seeman and Singer (2013). The triggering model generalizes the former two models by considering that agent i is influenced by a subset of its neighbors, S (called a trigger set), if any of the agents in S purchases the product. Note that this is a special case of our model when $\Gamma = 1$. For a general value of Γ, our model can be viewed as a *subset triggering model*. More precisely, agent i is influenced by its neighbors who belong to the trigger set S if at least Γ of them purchase the item. We next present several examples from this class of models that capture linear and nonlinear effects.

(1) Pairwise externality: Suppose $g_{S,i} = 0$ for all subsets $|S| > 1$ and $g_i, g_{j,i} \geq 0\ \forall i, j \in \mathcal{I}$ with $\Gamma = 1$. More specifically, we have

$$v_i(\alpha_i, \boldsymbol{\alpha}_{-i}) = \alpha_i \Big[g_i + \sum_{j \in \mathcal{I} \setminus \{i\}} \alpha_j g_{j,i} \Big]. \tag{3}$$

This case corresponds to a weighted bipartite graph $G(\mathcal{I}, \Delta_{\mathcal{I}}, E)$ with $\Delta_{\mathcal{I}} = \mathcal{I}$. This valuation function captures only the marginal externality of each neighbor and is linear and additive across neighbors. This type of model has been the focus of several earlier papers, such as Candogan et al. (2012), Bloch and Querou (2013), and Chen et al. (2011).

(2) Triple-wise externality: Suppose $g_{S,i} = 0$ for all subsets $|S| > 2$ and $g_i, g_{j,i}, g_{\{j,k\},i} \geq 0\ \forall i, j, k \in \mathcal{I}$. In particular, we have

$$v_i(\alpha_i, \boldsymbol{\alpha}_{-i}) = \alpha_i \Big[g_i + \sum_{j \in \mathcal{I} \setminus \{i\}} \alpha_j g_{j,i} + \sum_{j,k \in \mathcal{I} \setminus \{i\};\ j \neq k} \alpha_{j,k} g_{\{j,k\},i} \Big]. \tag{4}$$

In addition, we can have either of the following: (1) if $\Gamma = 1$, $\alpha_{j,k} = \max\{\alpha_j, \alpha_k\}$; (2) if $\Gamma = 2$, $\alpha_{j,k} = \alpha_j \alpha_k$. This valuation function is a special case of a supermodular utility model, in which the marginal effect of an additional neighbor increases with the set of existing influencers. Observe that these effects are characterized by externality terms that do not decompose by agents and are, hence, nonlinear with respect to subsets of neighbors.

Observe that, in this model, the externality effects of each subset is modeled separately. For example, consider a small network with three agents and $\Gamma = 2$, and suppose that all three agents decide to buy the item. The total network externalities of agents 2 and 3 on agent 1 are given by $g_{2,1} + g_{3,1} + g_{\{2,3\},1}$. The first two terms correspond to the pairwise effects (i.e., how agent j affects agent i on agent j's own), whereas the term $g_{\{2,3\},1}$ represents the additional externality of agents 2 and 3 *together* on agent 1. In particular, $g_{\{2,3\},1}$ does not incorporate the impact of its subsets ($g_{2,1}$ and $g_{3,1}$). The weighted bipartite graph $G(\mathcal{I}, \Delta_{\mathcal{I}}, E)$ for this example can be found in Figure A.1 (see Online Appendix A).

(3) Complete neighborhood triggering model: Suppose $g_{S,i} = 0$ for all $S \subset \mathcal{I} \setminus \{i\}$ except a single set $\mathcal{N}_i$, which represents all the neighbors of i.

$$v_i(\alpha_i, \boldsymbol{\alpha}_{-i}) = \alpha_i g_{\mathcal{N}_i,i} \max_{\substack{S' \subset \mathcal{N}_i \\ |S'| = \min\{\Gamma, |\mathcal{N}_i|\}}} \Big\{ \prod_{j \in S'} \alpha_j \Big\}.$$

In this model, an agent is influenced if and only if at least Γ of the agent's neighbors buy the item. By taking $\Gamma = 1$, we obtain a special case of a submodular influence model, in which only the first purchasing neighbor triggers an externality effect (and thereafter, the function does not increase with the number of purchasing neighbors). For other values of Γ, the function is neither submodular nor supermodular.

2.2. Pricing and Purchasing Model

Let the vector $\mathbf{p} \in \mathbf{P}$ denote the prices offered by the seller to all the agents. In particular, $p_i \in \mathbb{R}_+$ represents the price offered to agent i. Here, $\mathbf{P}$ is assumed to be a polyhedral set that represents the feasible pricing strategies, which possibly includes several business constraints. For example, the firm can adopt a discriminative pricing strategy in which each agent may potentially receive a different price; that is, $\mathbf{P} = \mathbb{R}_+^N$. In addition, one can restrict the values of these prices to lie between p_L and $p_U > p_L$; that is, $\mathbf{P} = \{\mathbf{p} \in \mathbb{R}_+^N | p_L \leq p_i \leq p_U\ \forall i\}$. A common pricing strategy is to adopt a single uniform price for all agents across the network. Here, $\mathbf{P} = \{\mathbf{p} \in \mathbb{R}_+^N | p_i = \bar{p}\ \forall i, \bar{p} \in \mathbb{R}_+\}$. Depending on the context, the firm can select appropriate business constraints on the pricing strategy. Finally, $\mathbf{P}$ can also incorporate specific constraints on network segmentation. For example, motivated by business practices, a particular segment of agents should be offered the same price. Alternatively, special members (e.g., loyal customers) should receive a lower price than regular customers.

Our goal is to develop a general and efficient optimal pricing method. For a given set of prices chosen by the seller, the agents in the network simultaneously choose their actions to maximize their utility (i.e., we consider a simultaneous game). We assume that the utility model of an agent is the total value minus the price:

$$u_i(\alpha_i, \boldsymbol{\alpha}_{-i}, p_i) = v_i(\alpha_i, \boldsymbol{\alpha}_{-i}) - \alpha_i p_i = \alpha_i\Big[\sum_{S\subset\mathcal{I}\setminus\{i\}} g_{S,i}\alpha_S - p_i\Big]. \tag{5}$$

If $\alpha_i = 1$, agent i purchases the item and derives a utility equal to $\sum_{S\subset\mathcal{I}\setminus\{i\}} g_{S,i}\alpha_S - p_i$, and if $\alpha_i = 0$, the agent does not purchase the item and derives zero utility. Each agent can buy at most one unit of the item and either fully purchases the item or does not purchase it at all.

Each agent is assumed to be rational and a utility maximizer. The utility-maximization problem of agent i can be written as follows:

$$\max_{\alpha_i\in\{0,1\}} u_i(\alpha_i, \boldsymbol{\alpha}_{-i}, p_i). \tag{6}$$

If the utility of an agent is exactly zero, the tie is broken, assuming this agent buys the item.

We assume that the seller is a profit maximizer with a linear manufacturing cost. The seller's problem is given by

$$\max_{\mathbf{p}\in\mathbf{P}} \sum_{i\in\mathcal{I}} \alpha_i\,(p_i - c), \tag{7}$$

where the vector $\boldsymbol{\alpha}$ represents the purchasing decisions of the agents obtained from problem (6) and c is the unit manufacturing cost. If agent i decides to buy the product at the offered price p_i, α_i is equal to one and the seller earns a profit of $p_i - c$. If agent i decides not to purchase the item, it incurs zero profit to the seller. The profit is denoted by Π.

We view the entire problem, called the pricing–purchasing game, as a two-stage Stackelberg game. First, the seller leads by choosing the prices $\mathbf{p}\in\mathbf{P}$ to be offered to the agents. Second, the agents follow by deciding whether to purchase the item, $\alpha_i\ \forall i\in\mathcal{I}$. We are interested in *subgame perfect equilibria* of this two-stage game (see, e.g., Fudenberg and Tirole 1991). For a fixed-price vector, the second stage equilibria, referred to as the *purchasing equilibria*, are defined by

$$\alpha_i^* \in \arg\max_{\alpha_i\in\{0,1\}} u_i\,(\alpha_i, \boldsymbol{\alpha}^*_{-i}, p_i) \quad \forall i\in\mathcal{I}.$$

We note that this definition is similar to the consumption equilibria of a divisible good in Candogan et al. (2012). However, in our case, the decision variables α_i are restricted to be binary so that agents cannot buy fractional amounts of the item. We also note that the two-stage problem is nonconvex as it includes terms of the form $\alpha_i p_i$ in the seller's objective and $\alpha_i\alpha_S$ in the objective functions of the agents (which are used as constraints in the seller's problem). In addition, the discrete nature of the purchasing decisions increases the complexity of the problem as it yields a nonconvex-integer program.

3. Purchasing Equilibria

In this section, we consider the second-stage game and show the existence of a pure Nash equilibrium (PNE) strategy given any price vector. We observe that there could be multiple pure Nash equilibria, but we characterize all these equilibria via a system of constraints using duality theory. We also identify a mild condition that allows us to focus on the purchasing equilibrium that is preferred by both the seller and the network of agents. We later show that our optimization formulation naturally induces this preferred purchasing equilibrium.

Theorem 1. *Consider the second-stage game played simultaneously by the network of agents.*

(1) *The second-stage game has at least one PNE for any given price vector* $\mathbf{p}$.

(2) *There exists a small* $\epsilon \ge 0$ *such that a price perturbation* $p_i - \epsilon\ \forall i\in\mathcal{I}$ *does not change any of the PNEs. In addition, it ensures that all agents in all PNEs strictly prefer one of the actions (buy or no buy).*

(3) *Among the multiple PNEs, there exists a unique Pareto-optimal PNE in which each agent's utility is at least as large as in any other PNE and is strictly higher for at least one agent. This implies that all agents who buy in any PNE will also buy in the Pareto-optimal PNE while deriving a higher utility.*

The proof can be found in Online Appendix B. The existence of a PNE follows from the fact that, for a given price vector, the second-stage game is of strategic complements (see, e.g., Jackson and Zenou 2014). Note that the first part of Theorem 1 guarantees the existence of a PNE but not necessarily its uniqueness. Consider the following simple example in which two distinct PNEs arise. Assume a network with two symmetric agents that mutually influence one another: $g_1 = g_2 = 2$ and $g_{21} = g_{12} = 1$. Consider the price vector $p_1 = p_2 = 2.5$. In this case, we have two PNEs: buy–buy and no buy–no buy. In other words, if player 1 buys, player 2 should buy, but if player 1 does not buy, player 2 will not either. Therefore, uniqueness is not guaranteed. More precisely, for any price strictly larger than three or strictly smaller than two, we have a unique equilibrium, but for any price between two and three, we have multiple equilibria. Nevertheless, for any price between two and $3-\epsilon$, the purchasing equilibrium is preferred by the agents as they each derive a positive utility from buying. In particular, for any price but three, ϵ can be set to zero, and for $p=3$, any small positive number will work. As a result, even though there exist multiple equilibria, by reducing the price by ϵ, the purchasing equilibria is strictly preferred by both agents. The purpose of ϵ, as can be noted from this example, is to ensure that agents with ties will buy without affecting other agents' decisions. Note that the value of ϵ can be taken very small so that it does not affect the revenue of the seller significantly.

In the last part of Theorem 1, we show that the seller's preferred equilibrium is unique and is also collectively preferred by all the agents. In particular, we show that among all PNEs, the preferred equilibrium has the property that any buyer in other PNEs will also buy in the preferred equilibrium. In Section 4, we demonstrate that the nature of the first-stage game always induces the purchasing equilibrium of interest (in this example, buy–buy), and hence, the rest of the paper focuses on this preferred equilibrium.

3.1. Characterization of the Purchasing Equilibria

The natural next step is to characterize the purchasing equilibria as a function of the prices, that is, to derive the functions $\alpha_i(\mathbf{p})\ \forall i \in \mathcal{I}$. This allows us to reduce the two-stage problem to a single optimization formulation, in which the only variables are the prices. In our setting, a closed-form expression for $\alpha_i(\mathbf{p})$ is not straightforward. Instead, by using duality theory, we characterize the set of constraints the equilibria should satisfy for any given price vector. We begin by making the following observation regarding the utility maximization problem of each agent.

Observation 1. Consider problem (6) for agent i under a given price vector $\mathbf{p}$. If other agents' decisions $\boldsymbol{\alpha}_{-i}$ are given, the problem of agent i has a tight LP relaxation.

In fact, for fixed values of $\mathbf{p}$ and $\boldsymbol{\alpha}_{-i}$, the subproblem faced by agent i happens to be a simple assignment problem. If the quantity $\left(\sum_{S\subset\mathcal{I}\setminus\{i\}} g_{S,i}\alpha_S - p_i\right)$ is positive, $\alpha_i^*=1$, and if this quantity is negative, $\alpha_i^*=0$. Finally, if this quantity is equal to zero, α_i^* can be any number in $[0,1]$ so that the agent is indifferent between buying and not buying. Therefore, the LP relaxation of the problem faced by agent i (for fixed values of $\mathbf{p}$ and $\boldsymbol{\alpha}_{-i}$) is integral.

Observation 1 allows us to transform problem (6) for agent i into a set of constraints by using duality theory. More specifically, this set of constraints consists of primal feasibility, dual feasibility, and strong duality conditions. For agent i, the constraints can be written as follows:

$$\text{Primal feasibility:} \quad 0 \le \alpha_i \le 1. \tag{8}$$

$$\text{Dual feasibility:} \quad y_i \ge \sum_{S\subset\mathcal{I}\setminus\{i\}} g_{S,i}\alpha_S - p_i, \tag{9}$$

$$y_i \ge 0. \tag{10}$$

$$\text{Strong duality:} \quad y_i = \alpha_i\Big(\sum_{S\subset\mathcal{I}\setminus\{i\}} g_{S,i}\alpha_S - p_i\Big). \tag{11}$$

Here, the variable y_i represents the dual variable of problem (6). Combining constraints (8)–(11) for all agents characterizes all the equilibria (mixed and pure) of the second-stage game as a function of the prices. To restrict our attention to the pure Nash equilibria (whose existence is guaranteed by Theorem 1), one can impose $\alpha_i \in \{0,1\}\ \forall i$. Observe that this characterization has reduced $N+1$ interconnected optimization problems to be compactly written as a single formulation. Note that the number of variables increases by N as we add a dual continuous variable for each agent.

4. Optimal Pricing: MIP Formulation

In this section, we use the existence and characterization of the PNEs to transform the two-stage problem into a single optimization formulation. This formulation happens to be a nonconvex-integer program but exhibits some interesting properties. We then reformulate the problem to arrive at an MIP with linear constraints.

We next formulate the pricing problem faced by the seller (denoted by Z) by incorporating the second-stage PNE characterized by constraints (9)–(11) and $\alpha_i \in \{0,1\}$ for each agent. The binary function defining α_S in Equation (2) is also included. The class of optimization problems with equilibrium constraints is referred to as mathematical program with equilibrium constraints and is well studied in the literature (see, e.g., Luo et al. 1996). The formulation is given by

$$\max_{\substack{\mathbf{p}\in\mathbf{P}\\ \mathbf{y},\boldsymbol{\alpha}}} \quad \sum_{i\in\mathcal{I}} \alpha_i(p_i - c) \tag{Z}$$

$$\text{s.t.} \quad \left.\begin{aligned} & y_i = \alpha_i\Big(\sum_{S\subset\mathcal{I}\setminus\{i\}} g_{S,i}\alpha_S - p_i\Big) \\ & y_i \ge \sum_{S\subset\mathcal{I}\setminus\{i\}} g_{S,i}\alpha_S - p_i \\ & y_i \ge 0 \\ & \alpha_i \in \{0,1\} \end{aligned}\right\} \quad \forall\, i \in \mathcal{I}$$

$$\alpha_S = \max_{\substack{S'\subset S\\ |S'|=\min\{\Gamma,|S|\}}} \Big\{\prod_{j\in S'} \alpha_j\Big\} \qquad \forall\, S \subset \mathcal{I}.$$

In addition to the presence of binary variables and the binary functions defining α_S, one can see that problem (Z) is nonlinear (and nonconvex) as it includes terms of the form $\alpha_i\alpha_S$ and $\alpha_i p_i$. Therefore, problem (Z) is not easily solvable by commercial solvers. We next show that, by introducing a few additional continuous variables, one can reformulate problem (Z) into an equivalent MIP with the same number of binary variables. To gain tractability, we consider valuation models in which the terms $g_{S,i}$ for large $|S|$ are set to zero. This motivates the following definition.

Definition 1. The K-wise utility model is a model in which $g_{S,i} = 0$ for all subsets $|S| > K - 1$.

Note that a larger value of K results in a more nonlinear utility model when compared with a smaller K (e.g., for $K = 2$, α_S is always linear, and for $K = 3$ it becomes quadratic). We next present the MIP for the K-wise utility model. We first define the following additional variables while also redefining the variable α_S:

$$z_i = \alpha_i p_i \qquad \forall\, i \in \mathcal{I}, \tag{12}$$

$$\alpha_S = \prod_{j\in S}\alpha_j \qquad \forall\, 1 < |S| < \Gamma + 2, S \subset \mathcal{I}, \tag{13}$$

$$\beta_S = \max_{S'\subset S, |S'|=\Gamma}\{\alpha_{S'}\} \qquad \forall\, \Gamma < |S| < K, S \subset \mathcal{I}, \tag{14}$$

$$\eta_{S,i} = \beta_S \alpha_i \qquad \forall\, i \notin S, \Gamma < |S| < K, S \subset \mathcal{I}, i \in \mathcal{I}. \tag{15}$$

The variables β_S and $\eta_{S,i}$ are defined for sets S satisfying $|S| > \Gamma$. The variables α_S are defined for sets S satisfying $|S| \le \Gamma + 1$. By using the binary nature of the variables and adding certain linear constraints, we can replace all nonlinear terms in problem (Z). This yields the following MIP denoted by Z-MIP. For simplicity of exposition, we present it for the case when $\Gamma = K - 1$ (and, hence, the β_S, $\eta_{S,i}$ variables are absent). The more general formulation can be found in Online Appendix C.

$$\max_{\substack{\mathbf{p}\in\mathbf{P}\\ \mathbf{y},\mathbf{z},\boldsymbol{\alpha}}} \sum_{i\in\mathcal{I}} (z_i - c\alpha_i) \tag{Z-MIP}$$

s.t.

$$\left.\begin{aligned} y_i &= \sum_{\substack{|S|<K\\ S\subset I\setminus\{i\}}} g_{S,i}\alpha_{S\cup i} - z_i \\ y_i &\ge \sum_{\substack{|S|<K\\ S\subset I\setminus\{i\}}} g_{S,i}\alpha_S - p_i \\ y_i &\ge 0 \end{aligned}\right\} \quad \forall\, i \in \mathcal{I} \tag{16}$$

$$\left.\begin{aligned} z_i &\ge 0 \\ z_i &\le p_i \\ z_i &\le \alpha_i p^{max} \\ z_i &\ge p_i - (1-\alpha_i)p^{max} \end{aligned}\right\} \quad \forall\, i \in \mathcal{I} \tag{17}$$

$$\left.\begin{aligned} \alpha_S &\ge 0 \\ \alpha_S &\le \alpha_{S\setminus\{i\}} \quad \forall\, i \in S \end{aligned}\right\} \quad \forall\, 1 < |S| < K+1, S \subset \mathcal{I} \tag{18}$$

$$\alpha_{S\cup\{i,j\}} \ge \alpha_{S\cup\{i\}} + \alpha_{S\cup\{j\}} - \alpha_S \quad \forall\, |S| < K-1, S \subset \mathcal{I}\setminus\{i,j\}, \{i\} \ne \{j\} \subset \mathcal{I} \tag{19}$$

$$\alpha_i \in \{0,1\} \qquad \forall\, i \in \mathcal{I} \tag{20}$$

$$\alpha_\emptyset = 1. \tag{21}$$

In this formulation, p^{max} denotes the maximal price allowed and is typically known from the context. For example, one can take $p^{max} = \max_i \left\{\sum_{S\subset I} g_{S,i}\right\}$ without affecting the problem at all because no agent would ever pay a price beyond this value. The set of constraints (17) aims to linearize and guarantee the definition of the variable z_i. The sets of constraints (18) and (19) linearize and ensure the correctness of the variable α_S. For example, constraint (19) for agents i and j and $S = \emptyset$ is given by $\alpha_{i,j} \ge \alpha_i + \alpha_j - 1$, which ensures $\alpha_{i,j} = \alpha_i\alpha_j$.

We note that in the Z-MIP formulation under the K-wise utility model, we have a total of (at most) $4N + \sum_{k=2}^{\Gamma+1}\binom{N}{k} + \sum_{k=\Gamma+1}^{K-1}\binom{N}{k}(1+N-k)$ variables ($4N$ for the α_i, $\mathbf{p}$, $\mathbf{y}$, and $\mathbf{z}$, the second term accounts for α_S, and the last term corresponds to β_S and $\eta_{S,i}$). However, only N variables are binary, and the remaining are all continuous. In particular, in the pairwise setting ($K = 2$), we have $4N + \binom{N}{2}$ variables, and in the triple-wise setting ($K = 3$), we have $4N + \binom{N}{2} + \binom{N}{3}$ variables if $\Gamma = 2$ and $4N + \binom{N}{2} + \binom{N}{3}(N-1)$ if $\Gamma = 1$. In other words, for small values of K (e.g., two or three), the number of variables is a small-degree polynomial in N.

We conclude that our problem of designing prices for selling an indivisible good to agents embedded in a social network can be formulated as an MIP. This MIP is equivalent to the two-stage nonconvex IP game with which we started. This formulation can be viewed as an operational tool to solve the optimal pricing problem (as we discuss in Section 6.2, the MIP formulation also holds when externalities can be negative). This is in contrast to previous approaches that proposed tailored algorithms for the problem in which one cannot easily incorporate business rules. However, solving an MIP may not be always feasible. If the size of the network is not very large, one can still solve it efficiently using commercial MIP solvers. Moreover, it is possible to solve the problem off-line (before launching a new product, for example) so that the running time may not be of the highest consideration. Potentially, one can also consider network-clustering methods to aggregate or coalesce several nodes to reduce the network size. If the network size is very large, one needs to find more efficient methods to solve Z-MIP. In the next section, we derive efficient optimal methods (polynomial in the number of agents) to solve the problem for two popular pricing strategies.

5. Efficient Optimal Algorithms

5.1. Discriminative Prices

We next consider the general pricing strategy in which the firm offers discriminative prices that potentially differ for each agent in the network; that is, $\mathbf{P} = \mathbb{R}_+^N$ in Z-MIP. This scenario is of interest in various practical settings in which the seller gathers the purchasing history of each potential buyer and the buyer's geographical location as well as other attributes. It can also be used by the seller to understand who the influential agents in the network are and what the maximal profit that can be achieved when using discriminative prices is. The prices can then be implemented by setting the same ticket price for everyone and sending out coupons with discriminative discounts. In fact, in practice, it often occurs that people receive different deals for the same item depending on the loyalty class, purchasing history, and geographical location. The method we propose aims to provide a systematic and automated way of finding the prices (equivalently, discounts) to offer to agents embedded in a social network based on their externalities to maximize the seller's profit.

As discussed, solving Z-MIP using an optimization solver may be impractical for large networks. We next show that solving the LP relaxation of Z-MIP yields the desired optimal integer solution. Consequently, one can solve the problem efficiently (polynomial in the number of agents and very fast in practice) and obtain an optimal solution even for large networks. Recall that the linearization of problem (Z) was possible because of the integrality of the decision variables. In other words, to reformulate problem (Z) into Z-MIP, the binary restriction was crucial. As a result, by introducing the new variables z_i, α_S, β_S, and $\eta_{S,i}$, one may potentially obtain fractional solutions that cannot be implemented in practice. However, the following theorem shows that the optimal solutions of Z-MIP can be identified using its continuous relaxation.

Theorem 2. *The optimal discriminative pricing solution of Z-MIP can be obtained efficiently (polynomial in the number of agents). In particular, problem Z-MIP admits a tight LP relaxation.*

The proof can be found in Online Appendix D. After solving the LP, we first order the agents in increasing order of α_i and then sequentially increase each agent's α_S value (with which $i \in S$) to the next agent's α_i value (equal to one after the last iteration). We do so carefully by maintaining the feasibility of the LP relaxation and without affecting other buyers' decisions or decreasing the seller's profit. This process is repeated until all agents who bought a fractional amount fully purchase the item. One can use this constructive argument or a solution approach, such as the simplex method, to arrive at the optimal extreme points that are guaranteed to be integer.

Theorem 2 suggests an efficient method to solve the problem that we formulated as a two-stage nonconvex-integer program. The LP-based method inherits all the complexity properties of linear programming and is, thus, applicable to large networks. Next, we derive some properties of the optimal solution. Interestingly, if we know the optimal set of buyers, the corresponding optimal prices can be obtained in closed form as summarized in the following observation.

Observation 2. Suppose T^* is the optimal set of buyers; that is, $T^* = \{i \in \mathcal{I} | \alpha_i^* = 1\}$. Let $S^*(T^*)$ be the sets for which $\alpha_S = 1$, $S \in S^*(T^*)$, obtained from Equation (2). Then, the optimized prices are

$$p_i^* = \begin{cases} \sum_{S \in S^*(T^*)\setminus\{i\}} g_{S,i} & \forall i \in T^* \\ p^{max} & \text{otherwise.} \end{cases} \tag{22}$$

In other words, the price of an agent who buys is the sum of the agent's own value (g_i) and a markup term corresponding to the externalities of the "buying" network on this agent. Membership of agent i into the buying class depends on the self-value g_i and on the externalities exerted by agent i on the network. The corresponding optimal profit is given by

$$\begin{aligned} \Pi^* &= \sum_{i \in T^*} \Big[\sum_{S \in S^*(T^*)\setminus\{i\}} g_{S,i} - c \Big] \\ &= \underbrace{\sum_{\{i \in T^* \,|\, g_i \geq c\}} (g_i - c)}_{\textit{Profit in the absence of network effects}} \\ &\quad + \underbrace{\sum_{\{i \in T^* \,|\, g_i < c\}} (g_i - c) + \sum_{\substack{i \in T^* \\ \emptyset \neq S \in S^*(T^*)\setminus\{i\}}} g_{S,i}}_{\textit{Incremental profit due to network effects}}. \end{aligned} \tag{23}$$

We separate the profit into two components. First is the profitable component of individual valuations, which the seller earns even in the absence of network externality effects. Second is the incremental profit resulting from network effects, which balances the seller's profit from unprofitable individual valuations with the revenue gain from network externalities.

Note that prices for nonbuyers are set to p^{max} to ensure that no agent will buy. The prices in Equation (22) are optimal as they result in the maximal profit that can be extracted by the seller given that agents in T^* are buying. That is, the seller extracts the full consumer surplus as expected because we consider a setting with complete information. One can see that T^* includes all agents in $\mathcal{I}$ whose individual valuations are profitable. In addition, some buyers may be offered a price below cost. One can view these agents as influencers who receive membership into the buying class because of their strong network externalities. On the flip side, the seller charges higher prices for strongly influenced agents. As a result, the seller taps into an additional source of profits by taking advantage of the network effects. The computational challenge lies in identifying the optimal set of buyers T^*. The LP approach presented in Theorem 2 allows efficiently finding T^* under the utility model (1).

We next use the closed-form solutions in Equations (22) and (23) to compare the prices and profit between the linear model in Equation (3) and the nonlinear models in Equation (4)a and (4)b. More precisely, we consider a setting in which $g_i \geq 0$ and $g_{j,i} \geq 0$ are the same across all three models. However, the $g_{\{j,k\},i}$ are set to zero in the linear model and are nonnegative in the other models. We denote the problem and the optimal solutions of models (3), (4)a, and (4)b with subscripts $(K = 2, \Gamma = 1)$, $(K = 3, \Gamma = 1)$, and $(K = 3, \Gamma = 2)$, respectively, and make the following observation.

Observation 3. The optimal solutions satisfy the following trends:

- $T^*_{(K=2,\Gamma=1)} \subset T^*_{(K=3,\Gamma=2)} \subset T^*_{(K=3,\Gamma=1)}$, where $T^* = \{i \in \mathcal{I} | \alpha^*_i = 1\}$;
- $p^*_{i,(K=2,\Gamma=1)} \leq p^*_{i,(K=3,\Gamma=2)}\ \forall\, i \in T^*_{(K=2,\Gamma=1)}$ and $p^*_{i,(K=3,\Gamma=2)} \leq p^*_{i,(K=3,\Gamma=1)}\ \forall\, i \in T^*_{(K=3,\Gamma=2)}$;
- $\Pi^*_{(K=2,\Gamma=1)} \leq \Pi^*_{(K=3,\Gamma=2)} \leq \Pi^*_{(K=3,\Gamma=1)}$, where Π^* denotes the optimal profit.

As we move from the linear model in Equation (3) to the nonlinear model (4), additional agents will buy. In addition, buyers will pay a higher price, hence inducing a larger seller's profit. As the value of Γ decreases, additional agents will buy. In particular, as the $g_{\{j,k\},i}$ term increases, it results in a larger number of agents buying the item. The seller can also charge higher prices and, consequently, earn a higher profit. The additional agents will further increase the profit. Interestingly, one can extend Observation 3 as a function of the degree of nonlinearity K and the threshold value Γ.

Corollary 1. *Consider the nonlinear utility model in Equation* (1) *for given K and Γ. Then*

- *The set of buyers, the optimal prices, and the seller's profit increase with K.*
- *The set of buyers, the optimal prices, and the seller's profit decrease with Γ.*

Corollary 1 allows us to understand the impact of both the degree of nonlinearity and the threshold value of our utility model on the optimal outcomes. Together with Equation (23), Corollary 1 highlights the importance of these parameters. In particular, the value of K affects the optimal prices and profit, and the value of Γ affects the active influential sets S. This result suggests that incorporating nonlinearity factors in the utility can significantly modify the pricing decisions.

5.2. Uniform Price

In this section, we consider the case in which the seller offers a uniform price across the network while incorporating externality effects. This scenario arises when the firm may not want to price discriminate because of fairness or ethical reasons and prefers to offer a uniform price. We observe that a similar result as in Theorem 2 for the setting with uniform pricing does not hold. In other words, by adding the (linear) uniform price constraint $p_1 = p_2 = \ldots = p_N$ to Z-MIP, the corresponding LP relaxation is no longer tight, and we obtain fractional solutions that cannot be implemented in practice. Geometrically, it means that incorporating such a constraint in the Z-MIP formulation is equivalent to adding a cut that violates the integrality of the extreme points of the feasible region. Therefore, we propose an alternative approach to optimally solve the problem by using an efficient algorithm based on iteratively solving the relaxed Z-MIP, which is an LP.

Theorem 3. *The optimal solution of Z-MIP for the case of a single uniform price can be obtained efficiently (polynomial in the number of agents) by applying Algorithm 1.*

We show the termination of Algorithm 1 in finite time and prove its correctness in Online Appendix E. At a high level, the procedure in Algorithm 1 iteratively reduces the size of the network by eliminating agents with low valuations (at least one such agent per iteration). As a result, it suffices to consider only a finite selection of prices (at least as high as cost) to identify the optimal uniform price.

Algorithm 1 (Procedure for Finding the Uniform Optimal Price)
Input: c, N, **and** G
Procedure

1. Set the iteration number to $t = 1$, solve the relaxed Z-MIP (an LP), and obtain the vector of discriminative prices $\mathbf{p}^{(1)}$.
2. Find the minimal discriminative price defined as $p^{(t)}_{min} = \max\{c, \min_{i \in \mathcal{I}} p^{(t)}_i\}$ and evaluate the objective function $\Pi^{(t)}$ with $p_i = p^{(t)}_{min}\ \forall\, i \in \mathcal{I}$ using formula (D.1).
3. Remove all agents who receive prices less than or equal to the minimal discriminative price from the network (including all their edges). If there are no more agents in the network, go to step 5. Otherwise, go to step 4.
4. Re-solve the relaxed Z-MIP for the reduced network and denote the output by $\mathbf{p}^{(t+1)}$. Set $t := t + 1$ and go to step 2.
5. The optimal uniform price is $p^{(\hat{t})}_{min}$, where $\hat{t} = \arg\max \Pi^{(t)}$, that is, the price that yields the highest profit.

6. Extensions

In this section, we consider two extensions of the models and results developed in Sections 3–5. First, we present two heuristic methods to solve the problem under discriminative prices. Second, we discuss the setting in which externalities among agents can be negative.

6.1. Heuristic Methods

In Section 5, we developed efficient optimal algorithms to solve the problem for discriminative prices and uniform price. Our solution approach is based on solving a (continuous) LP. Although these algorithms are efficient for most practical instances, they require the use of an optimization solver. In this section, we present two intuitive heuristic methods that are motivated by exploiting the insights drawn in Observation 2. Such approaches are transparent and easy to interpret but will generally yield a suboptimal solution. As noted in Observation 2, a key element of the optimal solution consists of identifying the optimal set of buyers T^*. Instead of solving an LP or enumerating all possible subsets (there are exponentially many), we propose two greedy approaches to construct the set of buyers. First, we present the greedy expansion procedure, in which we iteratively add to the set of buyers' agents who yield a positive marginal contribution to the profit. Second, we consider the greedy removal procedure, in which the initial set of buyers includes all agents and we iteratively remove the agent who decreases the seller's profit the most. The details of both procedures are reported in Algorithms 2 and 3, respectively. For simplicity of exposition, we focus on the utility model (3) that captures the individual network effects of neighbors only (i.e., $K = 2$ and $\Gamma = 1$). Note, however, that both heuristic methods easily extend to the more general utility model.

Algorithm 2 (Greedy Expansion Procedure)
Input: c, N, **and** G
Procedure

1. Assign all agents with $g_i \geq c$ to the set of buyers, which we denote by T_1^{GEP}. For all remaining agents $j \in N/T_1^{GEP}$, update g_j to $\tilde{g}_j = g_j + \sum_{i \in T_1^{GEP}} g_{i,j}$.
2. Assign all agents j with $\tilde{g}_j \geq c$ to the set of buyers denoted by T_2^{GEP}.
3. Repeat steps 1–2 until convergence. After this step, we call the set of buyers T_3^{GEP}. We are now left only with agents such that $\tilde{g}_i \geq c$.
4. For all remaining agents $i \in N/T_3^{GEP}$, compute the quantity $A_i = (\tilde{g}_i - c) + \sum_{k \in T_3^{GEP}} g_{i,k}$, if $A_i \geq 0$, add agent i to the set of buyers, which we denote by T_4^{GEP}. For all remaining agents $j \in N/T_4^{GEP}$, update $\tilde{g}_j$ to $\tilde{g}_j = g_j + \sum_{i \in T_4^{GEP}} g_{i,j}$.
5. Repeat step 4 until convergence. After this step, we call the set of buyers T_5^{GEP}.
6. For the remaining agents, we can use one of the following three options: (i) solve a smaller scale LP with the remaining agents, (ii) test adding each pair (or higher subsets) of agents to the set of buyers, or (iii) simply label the remaining agents as non-buyers.

Algorithm 3 (Greedy Removal Procedure)
Input: c, N, **and** G
Procedure

1. Assign all agents to the set of buyers, that is, $T_1^{GRP} = N$, and compute the vector of prices using equation (5.1) with T_1^{GRP} instead of T^*.
2. For each agent i, compute the quantity $B_i = (p_i - c) + \sum_{k \in T_1^{GRP}} g_{i,k}$. This quantity represents the contribution of including agent i to the set of buyers in the seller's profit.
3. If $B_i \geq 0, \forall i \in T_1^{GRP}$, the procedure terminates. Otherwise, remove the agent with the smallest (i.e., the most negative) value of B_i from the set of buyers. If several agents attain the smallest value, break ties randomly. Note that we can extend this step by considering simultaneously removing higher subsets of agents (e.g., pairs). After this step, we call the set of buyers T_3^{GRP}.
4. Repeat steps 1–3 (i.e., update the set of buyers to T_3^{GRP}, compute the prices and the quantity B_i for each i in T_3^{GRP}, and remove the agent with the smallest $B_i < 0$) until convergence. Note that in each step, we only need to update a small number of prices.

Both heuristic methods iteratively construct the set of buyers by exploiting the network externality structure among agents. In the greedy expansion procedure, we first ensure that all agents with a high self-value g_i purchase the item. We then use the network effects of such agents to identify new agents that will buy and assign them to the set of buyers. Next, we iteratively include to the set of buyers other agents who have a nonnegative marginal increase in the total seller's profit (captured by A_i). At this point, we are left with a smaller set of agents for which it is harder to determine if they belong to the set of buyers. Those agents have a negative marginal contribution, when we add exactly one of them to the set of buyers. Nevertheless, it is possible that adding several of them simultaneously is profitable. To solve this subproblem to optimality, one can solve a smaller LP or enumerate all possible subsets. Alternatively, one can simply assume that these agents are nonbuyers. In the greedy removal procedure, we first assign all agents to the set of buyers. We then iteratively remove the agent with the most negative marginal decrease to the seller's profit (captured by B_i). We stop the procedure when removing a single agent from the set of buyers does not increase the profit anymore. We next present parametric bounds on the profit performance of these heuristic methods. We denote by C_I the set of agents such that $g_i < c$ and by $|C_I|$ the number of such agents. We call g_{max} the maximal value of $g_{i,j}$ for agents in C_I, that is, $g_{max} = \max_{i \in C_I, j \in N} g_{i,j}$, and denote by $N_{I,max}$ the maximal number of neighbors for agents in C_I.

Proposition 1. *The greedy expansion and greedy removal procedures satisfy the following:*

(1) *Assume that the remaining agents in step 6 of Algorithm 2 are labeled as nonbuyers. In this case, the final set of buyers is T_5^{GEP} and satisfies $T_5^{GEP} \subseteq T^*$. In addition, the worst-case additive loss is $|C_I| N_{I,max} g_{max}$.*

(2) *The greedy removal procedure admits a worst-case additive loss of $|C_I|c$.*

Interestingly, the performance of both heuristic methods depends on the agents from the set C_I.

Specifically, the results of Proposition 1 can be shown by subtracting the contribution of all agents in C_I from the optimal profit (the formal proof is not reported for conciseness). Note that the first bound ($|C_I|N_{I,max}g_{max}$) represents the largest possible loss, which is computed by assuming that all agents in C_I collectively yield a positive profit. In most instances, the loss will be much smaller (for example, one can replace C_I by the set of remaining agents in step 6 of Algorithm 2). Similarly, the second bound ($|C_I|c$) corresponds to the worst-case of mistakenly including all agents in C_I. Note that when $C_I = \emptyset$, both greedy procedures yield the optimal solution as expected. These parametric bounds admit the following interesting implication. For the greedy expansion procedure, in the worst case, we miss all agents with $g_i < c$, whereas, in the greedy removal procedure, in the worst case, we mistakenly include all agents with $g_i < c$. This implies that, when one heuristic method will not perform well, the other will and vice versa. Interestingly, one can show that the greedy removal procedure is optimal for the special case of supermodular valuation functions.

In summary, we have exploited the structure of the problem to propose two greedy heuristic procedures. As mentioned, one can also use the greedy expansion procedure to significantly reduce the size of the LP to compute the optimal solution, and this method can also be used as a subroutine in Algorithm 1 instead of the LP.

6.2. Setting with Negative Externalities

In the previous sections as well as in several papers in the literature, the focus has mainly been on settings with nonnegative externalities. One exception is the recent work in Cao et al. (2017) in which the authors take an algorithmic approach to solve the iterative pricing problem with negative externalities. They show that the problem is NP-hard and propose a two-approximation algorithm. The problem considered in Cao et al. (2017) is different than ours as the seller can post an iterative list of uniform prices for all the agents (multiple selling rounds). In addition, their utility model is a special case of our model (they consider pairwise interactions terms that are all negative). In many practical applications, the externalities among agents in the network can be both positive and negative (e.g., negative reviews on a product sold via an online platform). In this section, we discuss how our results change for a setting in which externalities can be either positive or negative.[2] First, the existence of a PNE for the second-stage game is not always guaranteed when externalities can be negative (and, therefore, Theorem 1 does not hold in general). Nevertheless, the characterization method of the purchasing equilibria still applies. In particular, the Z-MIP formulation presented in Section 4 remains the same. The main difference is that when externalities can take negative values, it is possible that the MIP is infeasible. In this case, it means that there is no PNE for the purchasing game under any price vector. By solving the Z-MIP problem, we can then determine whether a PNE exists, and if it does, we can compute the most profitable solution for the seller. We summarize this result in the following corollary.

Corollary 2. *Consider a setting with general externalities. If Z-MIP is infeasible, it means that there is no PNE under any price vector. Otherwise, its optimal solution leads to the optimal price vector and to the resulting purchasing PNE. This optimal solution corresponds to the most profitable outcome for the seller.*

The implication of Corollary 2 is twofold: (1) the Z-MIP formulation allows us to determine whether a PNE exists for the network of agents under any price vector, and (2) if it does, it can compute the optimal outcome for the seller. Computing an equilibrium under negative externalities is a hard problem (Cao et al. (2017) show that a similar version of this problem is NP-hard). As a result, the Z-MIP formulation we introduce in this paper allows us to solve the problem for a wide range of utility models under either positive or negative externalities. Unfortunately, the efficient optimal approach presented in Section 5.1 uses the nonnegativity of the externalities to ensure the integrality of the formulation. We then propose either to solve Z-MIP directly or to use one of the heuristic methods presented in Section 6.1. Note, however, that the parametric bound in Proposition 1 holds only for the setting with nonnegative externalities.

7. Price Incentives to Guarantee Influence

So far, we have assumed that network externality effects always materialize as long as agents purchase the item. This assumption is not realistic in many practical settings. After purchasing an item, it is sometimes not entirely natural to exert network externalities unless one takes some effort to do so. This, for example, could be by writing a review, endorsing the item on social media, or at the very least announcing the purchase. However, to the best of our knowledge, most previous work imposes the assumption that purchasing is equivalent to influencing (i.e., exerting network externality effects) with the exception of Arthur et al. (2009). In the latter, the authors study a cash-back setting in which the seller offers an exogenous uniform cash reward to any recommender if the recommender influenced at least one friend to purchase the item. We study a similar model in the context of purchasing equilibrium and the optimization framework proposed in this paper.

Consider a setting in which the seller offers both a price and a discount (also referred to as incentive) to each agent. If the agent decides to purchase the item, the agent can claim the discount in return for influence actions, such as liking the product in online platforms or writing a review. Using such a model, the seller can now ensure the externalities so that network effects are guaranteed to occur. In the previous setting in which externalities were assumed to always occur, the actual profits may be far from the value predicted by the optimization. In fact, we show via a computational example in Section 8 that even if a few agents do not exert the agents' externality effects, it can significantly reduce the seller's profit. We next extend our model and results to this more general setting.

7.1. Model

For simplicity, we focus on the utility model (3) that captures individual externalities of neighbors only. We consider a model with a continuum of actions to influence neighbors. Let $t_i \geq 0$ denote the utility of the maximal effort needed by agent i to claim the entire discount offered by the seller. If agent i decides to purchase the item, we assume that $\gamma_i t_i$ is the effort required by agent i to claim a fraction γ_i of the discount, and $0 \leq \gamma_i \leq 1$. We view t_i as the *influence cost* of agent i and the variable γ_i as the *externality intensity* chosen by agent i. The parameter t_i can be estimated from historical data, such as past purchases and number of reviews written. For a given price vector $\mathbf{p}$ and discount vector $\mathbf{d}$, we extend the utility function of agent i in Equation (5) as follows:

$$u_i(\alpha_i, \gamma_i, \boldsymbol{\alpha}_{-i}, \boldsymbol{\gamma}_{-i}, p_i, d_i) = \alpha_i\Big(g_i + \sum_{j \in \mathcal{I}\setminus\{i\}} \gamma_j g_{ji} - p_i\Big) + \gamma_i(d_i - t_i),$$

where α_i is the binary purchasing decision of agent i and $\gamma_i \leq \alpha_i$. If agent i does not purchase the item, $\alpha_i = 0$ and $\gamma_i = 0$. In other words, the constraint $\gamma_i \leq \alpha_i$ captures the fact that only buyers can exert externalities on friends. However, if agent i purchases the item, then $\alpha_i = 1$ and γ_i can be any number in $[0, 1]$ as chosen by agent i. Here, $\boldsymbol{\alpha}_{-i}$ and $\boldsymbol{\gamma}_{-i}$ are the decisions of all other agents but i. Similarly to problem (6), the utility maximization problem for agent i is given by

$$\begin{aligned} \max_{\alpha_i, \gamma_i} \quad & u_i(\alpha_i, \gamma_i, \boldsymbol{\alpha}_{-i}, \boldsymbol{\gamma}_{-i}, p_i, d_i) \\ \text{s.t.} \quad & 0 \leq \gamma_i \leq \alpha_i \\ & \alpha_i \in \{0, 1\}. \end{aligned} \tag{24}$$

In a similar way as problem (7), the seller's profit-maximization problem can be written as

$$\max_{\mathbf{p}, \mathbf{d} \in \mathbf{P}} \sum_{i \in \mathcal{I}} \Big[\alpha_i (p_i - c) - \gamma_i d_i\Big]. \tag{25}$$

Here, the seller's decision variables are $\mathbf{p}$ and $\mathbf{d}$, which are two vectors of prices and discounts with a component potentially different for each agent. As before, these vectors can be chosen according to different strategies. For example, one can consider a fully discriminative, a uniform pricing strategy, or—more generally—a hybrid model in which the regular price is uniform across the network (i.e., $p_i = p_j$), but the discounts are tailored to the various agents. This hybrid setting corresponds to a common practice of online sellers who offer a standard posted price but design personalized discounts for different classes of customers (sent via targeted coupons). The variables α_i and γ_i are decided according to each agent's utility maximization problem given in Equation (24). If agent i decides to buy the product, then the seller incurs a profit of $p_i - \gamma_i d_i - c$.

In the special case in which $\alpha_i = \gamma_i$ and $t_i = 0 \ \forall i \in \mathcal{I}$, we recover our previous model. In addition, by adding the constraint $\gamma_i \in \{0, 1\}$, we have an interesting setting in which each agent can only buy at two different prices: a full price p_i (that does not require any action) and a discounted price $p_i - d_i$ that requires an action to influence. Note that one can easily extend the model in this section to more than two prices to incorporate a finite set of different actions specified by the seller.

7.2. Results

Our goal is to extend our results to this general setting. We next show that for any given prices and discounts there exists a PNE for the second-stage game.

Theorem 4. *The second-stage game has at least one pure Nash equilibrium for any given vectors of prices* $\mathbf{p}$ *and discounts* $\mathbf{d}$ *chosen by the seller. A small perturbation in prices and discounts results in a Pareto-optimal PNE that is preferred by both the seller and the network of agents.*

The proof of Theorem 4 is not presented for conciseness and is of similar nature as the proof of Theorem 1. In this case, a PNE is defined by restricting the purchasing decisions α_i to be zero or one. Nevertheless, we note that there always exists an equilibrium in which the variables γ_i are also all integer. More precisely, if $d_i - t_i > 0$ (recall that prices and discounts are given), γ_i can be set to one, and otherwise, $\gamma_i = 0$. As a result, there exists a PNE with γ_i integer. Note that a result similar to Observation 1 still holds, and hence, one can characterize the equilibria (mixed and pure) as a set of constraints in which the binary variables are relaxed to be continuous. In this case, one can transform subproblem (24) to a set of constraints using duality theory:

$$\text{Primal feasibility:} \quad 0 \leq \alpha_i \leq 1, \tag{26}$$

$$0 \leq \gamma_i \leq \alpha_i. \tag{27}$$

$$\text{Dual feasibility:} \quad y_i - w_i \geq g_i + \sum_{j \in \mathcal{I}\setminus\{i\}} \gamma_j g_{ji} - p_i, \tag{28}$$

$$w_i \geq d_i - t_i, \tag{29}$$

$$y_i, w_i \geq 0. \tag{30}$$

$$\text{Strong duality:} \quad y_i = \alpha_i\Big(g_i + \sum_{j \in \mathcal{I}\setminus\{i\}} \gamma_j g_{ji} - p_i\Big) + \gamma_i(d_i - t_i). \tag{31}$$

In this case, we have two continuous dual variables y_i and w_i, together with two dual feasibility constraints for each agent i. Similar to the earlier setting, we impose α_i to be binary for all $i \in \mathcal{I}$ to restrict to pure equilibria. We can then formulate the optimal pricing problem, similar to problem Z, that maximizes the profit given in Equation (25) with the equilibrium constraints (26)–(31):

$$\max_{\substack{\mathbf{p},\mathbf{d}\in\mathbf{P}\\ \mathbf{y},\mathbf{w},\boldsymbol{\alpha},\boldsymbol{\gamma}}} \sum_{i\in\mathcal{I}} \left[\alpha_i(p_i - c) - \gamma_i d_i\right]$$
$$\text{s.t.} \quad \text{constraints (27)–(31)}, \alpha_i \in \{0,1\} \quad \forall\, i \in \mathcal{I}.$$

We denote this problem by Z*i*, where *i* represents the model with incentives to guarantee influence of the present section. We make the following observation.

Observation 4. Every optimal solution of problem Z*i* satisfies $d_i \le t_i$.

This follows from the fact that the seller can reduce d_i to be equal to t_i while maintaining feasibility and strictly increasing the objective. This implies that constraint (29) is redundant in the optimal pricing problem. By using constraints (28)–(30), one can always assign $w_i = 0$ while maintaining feasibility without altering the objective. This observation allows us to simplify problem Z*i* by removing all dual variables $w_i\ \forall i \in \mathcal{I}$.

Proposition 2. *Problem* Z*i* *admits a tight continuous relaxation. Moreover, there always exists an optimal solution to problem* Z*i* *in which all variables* γ*s are also integer.*

The second result in Proposition 2 is interesting as it implies that, even though the seller allows for a continuum of influence actions, the buyer would either fully influence or not influence at all. As a result, this is equivalent to the setting in which the seller offers only two options: a full price p_i and a discounted price $p_i - d_i$ in exchange for a specific action to influence.

Problem Z*i* has nonlinearities of the form $\alpha_i\gamma_j$, $\alpha_i p_i$, and $\gamma_i d_i$. Using the discreteness of α_i and γ_i from Proposition 2, one can transform problem Z*i* into the following MIP, denoted by Z*i*-MIP:

$$\max_{\substack{\mathbf{p},\mathbf{d}\in\mathbf{P}\\ \mathbf{y},\mathbf{z},\mathbf{z^d},\mathbf{x},\boldsymbol{\alpha},\boldsymbol{\gamma}}} \sum_{i\in\mathcal{I}} (z_i - z_i^d - c\alpha_i) \qquad (\text{Z}i\text{-MIP})$$

s.t.

$$\left.\begin{aligned} y_i &= \left(\alpha_i g_i + \sum_{j\in\mathcal{I}\setminus\{i\}} x_{ji} g_{ji} - z_i\right) + (z_i^d - \gamma_i t_i)\\ y_i &\ge g_i + \sum_{j\in\mathcal{I}\setminus\{i\}} \gamma_j g_{ji} - p_i\\ \gamma_i &\le \alpha_i\\ y_i &\ge 0 \end{aligned}\right\} \quad \forall\, i\in\mathcal{I} \qquad (32)$$

$$\left.\begin{aligned} z_i, z_i^d &\ge 0\\ z_i &\le p_i\\ z_i &\le \alpha_i p^{max}\\ z_i &\ge p_i - (1-\alpha_i)p^{max}\\ z_i^d &\le d_i\\ z_i^d &\le \gamma_i p^{max}\\ z_i^d &\ge d_i - (1-\gamma_i)p^{max} \end{aligned}\right\} \quad \forall\, i\in\mathcal{I} \qquad (33)$$

$$\left.\begin{aligned} x_{ji} &\ge 0\\ x_{ji} &\le \alpha_i\\ x_{ji} &\le \gamma_j\\ x_{ji} &\ge \alpha_i + \gamma_j - 1 \end{aligned}\right\} \quad \forall\, i \ne j\in\mathcal{I} \qquad (34)$$

$$\alpha_i, \gamma_i \in \{0,1\} \qquad \forall\, i\in\mathcal{I}. \qquad (35)$$

Note that we removed the dual variables w_i using Observation 4. We conclude that the problem of designing prices and incentives for selling an indivisible item to agents in a social network can be formulated as an MIP. For the case of discriminative prices and discounts, that is, when $\mathbf{P} = \mathbb{R}_+^N \text{x}\, \mathbb{R}_+^N$, we next show a similar result as Theorem 2.

Theorem 5. *The optimal discriminative pricing solution of* Z*i-MIP can be obtained efficiently (polynomial in the number of agents). In particular, problem* Z*i-MIP admits a tight LP relaxation.*

The main idea behind the proof is composed of the following two steps. First, fix the values of γ_i, z_i^d and proceed in the same fashion as in Theorem 2 to construct a solution with α_i integer $\forall i \in \mathcal{I}$. Second, with the integer $\boldsymbol{\alpha}$ values obtained from the previous step, one can show that the objective does not decrease by modifying any component of $\boldsymbol{\gamma}$ to one by appropriately altering the prices of the neighbors so that their actions do not change as in Proposition 2.

In comparison with problem Z-MIP with a single price for each agent, problem Z*i*-MIP yields potentially a lower profit for the seller. However, this profit is guaranteed, whereas, in the previous case, the estimated profit can be far from the realized value if people fail to exert their externalities on neighbors (i.e., the model is misspecified). The difference in profits between both settings can be viewed as the price paid by the seller to guarantee network externalities and can be computed efficiently by solving both settings.

8. Computational Experiments

In this section, we present computational experiments on simple networks to draw qualitative insights and to compare various pricing strategies, including the richer model with incentives from Section 7. We consider a network with $N = 10$ agents and a utility model with $K = 2$ and $\Gamma = 1$.

8.1. Value of Incorporating Network Externalities

In Figure 1, we plot the optimal prices offered by the seller under discriminative and uniform pricing strategies, both with and without network externalities. The circles around the markers, whenever present, depict the fact that the agent decided not to purchase the item at the offered price (agents 7–9 for uniform price with network externalities). In this instance, each agent is connected to exactly three other agents, and we use $g_{j,i} = 1.25$ for any connected edge, $g_i = 3$, and $c = 2$.[3]

We observe that incorporating the positive externalities among the agents allows the seller to earn higher profits. In this particular example, the total profits are equal to 46.25 (discriminative prices) and 24.5 (uniform price) for the case with network externalities. In the case without network externalities, the profits under both uniform and discriminative prices are equal to 10. This result is expected as each agent's willingness to pay increases as the agent's neighbors positively affect the agent. The seller can, therefore, charge higher prices and increase its profits. Figure 1 also shows the added benefit of using a discriminative pricing strategy relative to a uniform price. When the firm has the additional flexibility to price discriminate and offers a different price to each agent in the network, the total profit can increase significantly.

8.2. Pricing an Influencer

In Figure 2, we present an example in which it is beneficial for the seller to earn a negative profit ($p_i < c$) from an influential agent to extract significant positive profits from the agent's neighbors. In particular, we consider a network in which agent 5 is a very influential player with $g_{5,5}$ being very low (0.075) while $g_{5,j}$ is sufficiently high (1.38) for the four agents that agent 5 influences. Here, $g_{i,j} = 0.75$ for any other connected edge, $g_i = 1.5R\ \forall i \neq 5$, where $R = U[1, 2]$ (a single independent and identically distributed (i.i.d.) instance is drawn) and $c = 2$. In this example, the optimal discriminative price for agent 5 happens to be below cost. This illustrates the fact that agent 5 has an influential position in the network, and therefore, the seller should strongly incentivize this agent. In particular, the optimal algorithm identifies this pattern and captures the fact that it is profitable to offer a low price to agent 5. This way, the seller loses a small amount of money from the influential agent but can extract higher profits from others. We now consider an alternative strategy in which the seller decides to remove agent 5 from the network because of agent 5's low valuation (this is equivalent to offering a very large price to agent 5). In this case, all the optimal prices are decreased, and the overall profit drops from 63.52 to 55.5. As a result, one can increase the profit by 14.5% by including agent 5.

8.3. Value of Incorporating Incentives That Guarantee Influence

In Figure 3, we compare the optimal solution for discriminative prices to the more general model from Section 7 in which the seller offers a uniform regular price ($p = 4$) and designs discriminative discounts in exchange for an action. As discussed, if the seller does not incentivize the agents to influence, it is possible that some of them would not exert their externalities. The goal of this experiment is to quantify the impact of having incentives. In this instance, each agent is connected to three other agents with $g_{j,i} = 0.75$ for any connected edge and $g_i = 1.5R$, where $R = U[1, 4.5]$ (a single i.i.d. instance is drawn). We assume that $t_i = U[0, 1]\ \forall\, i \neq 1$ (single i.i.d. drawn), $t_1 = 6.9$, and $c = 1$.

Figure 1. (Color online) Value of Incorporating Network Externalities for Discriminative and Uniform Pricing Strategies

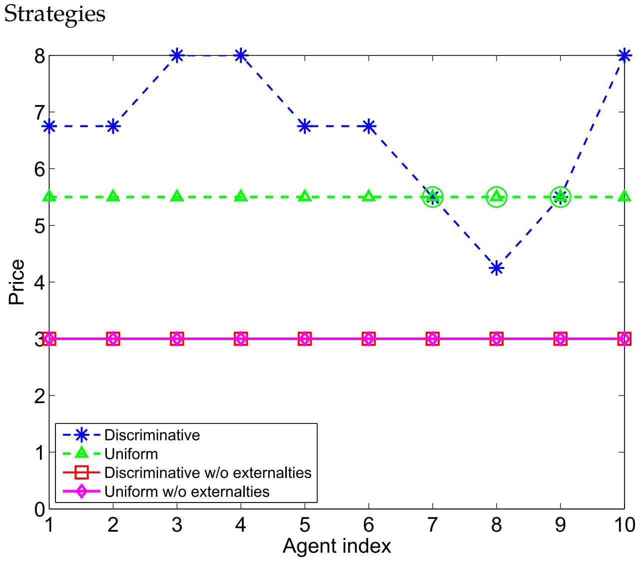
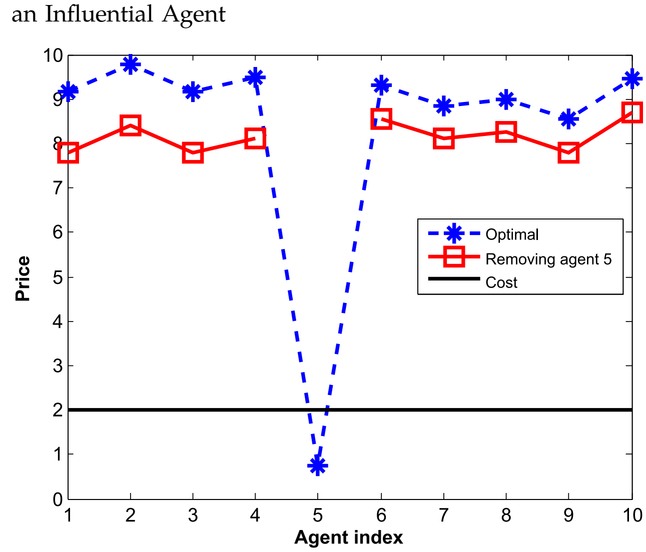


Figure 2. (Color online) Centrality Effect: Losing Money on an Influential Agent



We observe that the profit using the model without incentives is equal to 27.15. This profit is not guaranteed because some agents may not influence their peers (i.e., the model may be misspecified). In particular, in this example, suppose agents 5 and 10 who buy at the full price do not influence their neighbors. Agent 1 ends up not purchasing the item and, consequently, does not influence agent 1's neighbors either. Finally, it happens that only agents 2, 5, and 10 buy the item, yielding a profit of nine as opposed to 27.15. As a result, the earlier model predicts a profit value that is significantly higher than the realized profit. On the other hand, in the model with incentives that guarantee influence, the total profit is equal to 20.85. In this case, agent 1 does not purchase the item, and agents 5 and 10 do not influence anyone, but other agents do. Observe that this is lower than 27.15 but significantly higher than nine. Therefore, the model with incentives provides the seller with the flexibility of using prices together with incentives that result in a higher degree of confidence on the predicted profit value.

8.4. Symmetric Agents with Asymmetric Incentives

In Figure 4, we present a setting with symmetric agents who receive asymmetric incentives to influence their neighbors. In this instance, each agent has the same number of neighbors and the same self- and cross-valuations. In particular, we consider a complete graph with $g_i = 1.3$ and $g_{i,j} = 0.3$, a cost to influence $t_i = 2.2$, and $c = 0.2$. We compute the optimal discriminative prices, which happen to be three for everyone, and compare with the case in which the seller designs incentives to guarantee influence by offering two prices (using problem *Zi*-MIP). Interestingly, the optimal solution for the model with incentives is not symmetric despite the fact that all agents are homogenous.

Figure 3. (Color online) Value of Incorporating Incentives That Guarantee Influence

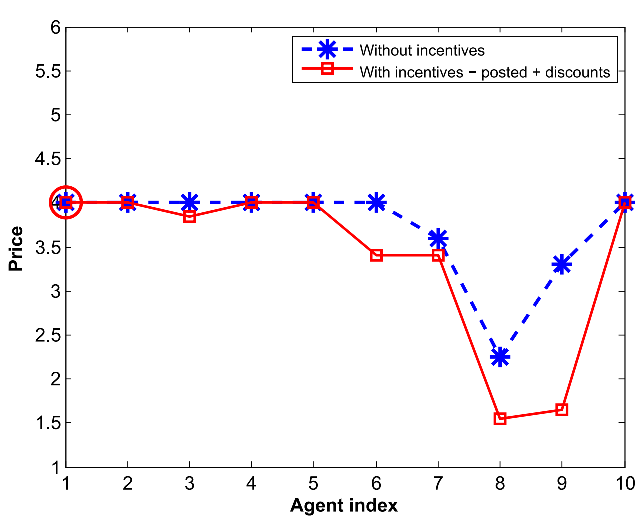
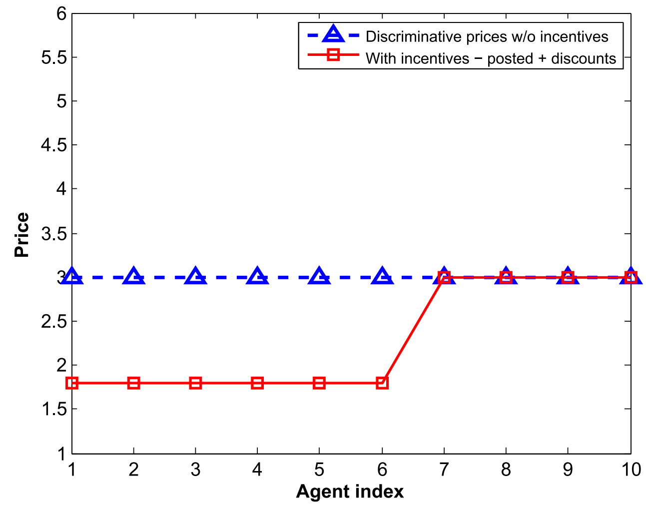


Figure 4. (Color online) Symmetric Graph with Asymmetric Incentives: With and Without Incentives

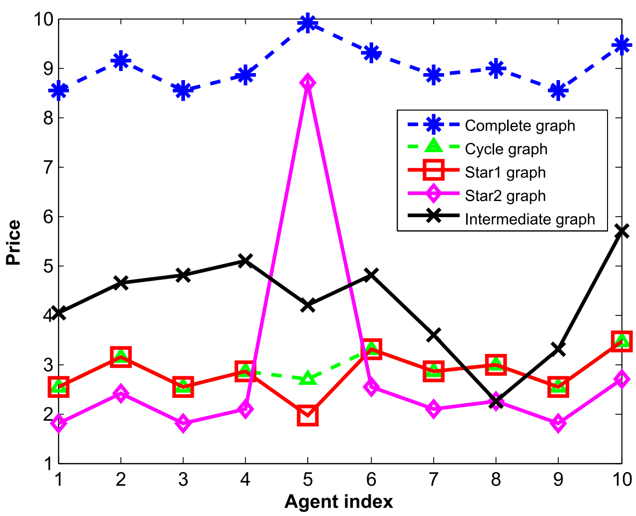


Indeed, it is sufficient to incentivize any six out of 10 agents in the network (no matter which group of six). These six agents receive a targeted discount to exert network effects on their peers that purchase at the full price.

8.5. Effect of Network Topology on Optimal Prices

In Figure 5, we consider different network topologies and compare the optimal discriminative prices and the corresponding profits. In all the scenarios, $g_i = 1.5R$, where $R = U[1, 2]$ (a single i.i.d. instance is drawn), $g_{i,j} = 0.75$ when agent i influences agent j and zero otherwise, and $c = 2$. For each network topology, we solve the optimal discriminative prices using the Z-MIP relaxation. We plot the optimal price vector for the different networks in Figure 5. We observe that, in our example, all agents purchase the item. In the complete graph, all nodes are connected to each other, and hence, the profits are the highest (70.15). In the intermediate topology in which each agent has three neighbors, the total profits are equal to 22.45. The cycle graph is a network in which the nodes are connected in a circular fashion and each agent has one ingoing and one outgoing edge. In this case, the total profits are equal to 8.95. Star 1 and star 2 are star graphs with a central agent being agent 5. In star 1, agent 5 influences all other agents, and in star 2, agent 5 is influenced by all others. In both cases, the profits are equal to 8.2 as the total valuations in the system are identical. In star 1, agent 5 receives a small discount to influence so that the prices offered to other buyers are slightly higher. In star 2, the prices of all agents but agent 5 are slightly lower so that the seller can charge a high price to agent 5. As we can see, the prices and profits increase with the number of edges in the graph. Indeed, each additional edge corresponds to an agent increasing another agent's

Figure 5. (Color online) Optimal Prices for Various Network Topologies



willingness to pay, and therefore, the more connected the graph is, the higher the profit.

9. Conclusions

In this paper, we study an optimal pricing model for a firm that sells an indivisible good to agents embedded in a social network. We assume that agents interact and positively influence each other's purchasing decisions (via network externalities). We propose a broad class of nonlinear utility models that explicitly capture externalities from subsets of agents (communities or groups) and allow a threshold on the number of agents needed to trigger the externality effect.

We model the problem as a two-stage game and reformulate it as an MIP with linear constraints. We view this MIP as an operational pricing tool that holds for general externalities (positive or negative) and that can incorporate various pricing business rules. For the case of discriminative and uniform pricing strategies (under positive externalities), we present efficient methods to optimally solve the MIP using its LP relaxation. We observe that the price of a buyer in the optimal discriminative solution can be expressed as the sum of its own value and a markup term corresponding to externalities from the network of agents who buy the item. The gain from network externalities comes from two types of customers: high-valued customers and low-valued customers who are influential and can sometimes be offered a price below cost. In addition, when comparing linear to nonlinear utility models, we show that, as we move from a linear model to a nonlinear one, additional agents will buy and buyers will pay a higher price, and hence, this induces higher profits. We also convey that a larger threshold on the minimum number of neighbors results in a smaller number of buyers and decreases the seller's profit. Hence, our analysis suggests that incorporating nonlinearity effects in the utility model significantly modifies the pricing decisions.

We extend our pricing model and results to the case in which the seller can design both prices and incentives to guarantee influence. This extension is important as, in general, agents that buy do not necessarily exert network externalities on their peers. The seller can use incentives in exchange for an action, such as a wall post or a review to guarantee network externality effects. Finally, we present computational experiments to highlight the benefits of incorporating network externalities and to compare the different pricing strategies.

Acknowledgments

The authors thank the department editor (Morris Cohen), the associate editor, and the two anonymous referees for their insightful comments, which have helped improve this paper. The authors would also thank Itai Ashlagi, Adam Elmachtoub, Markus Ettl, Georgia Perakis, and Velibor V. Mišić for their valuable feedback that helped improve this paper. Finally, the first author thanks IBM Research as this work would not have been possible without an internship at the IBM T. J. Watson Research Center during the summer of 2012.

Endnotes

[1] In the context of this paper, the term "influence" refers to exerting network externalities (or network effects).

[2] We refer to *general externalities* to describe the setting that includes both positive and negative externalities among agents in the network.

[3] In Figure 1, the exact network structure is such that agents 2, 3, and 10 have four influencers; agents 1, 2, 5, and 6 have three influencers; agents 7 and 9 have two influencers; and finally, agent 3 has one influencer.

References

Akhlaghpour H, Ghodsi M, Haghpanah N, Mirrokni V, Mahini H, Nikzad A (2010) *Optimal Iterative Pricing Over Social Networks*, Lecture Notes in Computer Science, vol. 6484 (Springer, Berlin, Heidelberg), 415–423.

Andreasen AR (1994) Social marketing: Its definition and domain. *J. Public Policy Marketing* 13(1):108–114.

Aral S, Walker D (2011) Creating social contagion through viral product design: A randomized trial of peer influence in networks. *Management Sci.* 57(9):1623–1639.

Arthur D, Motwani R, Aneesh S, Xu Y (2009) Pricing strategies for viral marketing on social networks. *Proc. 5th Internat. Workshop Internet Network Econom.* (Springer, Berlin, Heidelberg), 101–112.

Ballester C, Calvó-Armengol A, Zenou Y (2006) Who's who in networks. Wanted: The key player. *Econometrica* 74(5):1403–1417.

Banerji A, Dutta B (2009) Local network externalities and market segmentation. *Internat. J. Indust. Organ.* 27(5):605–614.

Basu A, Mazumdar T, Raj S (2003) Indirect network externality effects on product attributes. *Marketing Sci.* 22(2):209–221.

Bertsimas D, Weismantel R (2005) *Optimization Over Integers*, vol. 13 (Dynamic Ideas, Belmont, MA).

Bloch F, Querou N (2013) Pricing in social networks. *Games Econom. Behav.* 80:243–261.

Borodin A, Filmus Y, Oren J (2010) Threshold models for competitive influence in social networks. *Internat. Workshop Internet Network Econom.* (Springer, Berlin, Heidelberg), 539–550.

Candogan O, Bimpikis K, Ozdaglar A (2012) Optimal pricing in networks with externalities. *Oper. Res.* 60(4):883–905.

Cao Z, Chen X, Hu X, Wang C (2017) Approximation algorithms for pricing with negative network externalities. *J. Combin. Optim.* 33(2):681–712.

Chen W, Wang C, Wang Y (2010) Scalable influence maximization for prevalent viral marketing in large-scale social networks. *Proc. 16th ACM SIGKDD Internat. Conf. Knowledge Discovery Data Mining* (ACM, New York), 1029–1038.

Chen W, Lu P, Sun X, Tang B, Wang Y, Zhu ZA (2011) Optimal pricing in social networks with incomplete information. *Internat. Workshop Internet Network Econom.* (Springer, Berlin, Heidelberg), 49–60.

Domingos P, Richardson M (2001) Mining the network value of customers. *Proc. 7th Conf. Knowledge Discovery Data Mining* (ACM, New York), 57–66.

Farrell J, Saloner G (1985) Standardization, compatibility and innovation. *RAND J. Econom.* 16(1):70–83.

Fudenberg D, Tirole J (1991) *Game Theory* (MIT Press, Cambridge, MA).

Goldenberg J, Libai B, Muller E (2001) Talk of the network: A complex systems look at the underlying process of word-of-mouth. *Marketing Lett.* 12(3):211–223.

Gunnec D, Raghavan S (2017) Integrating social network effects in the share-of-choice problem. *Decision Sci.* 48(6):1098–1131.

Hartline J, Mirrokni V, Sundararajan M (2008) Optimal marketing strategies over social networks. *Proc. 17th Internat. Conf. World Wide Web* (ACM New York), 189–198.

Iyengar R, den Bulte CV, Valente TW (2011) Opinion leadership and social contagion in new product diffusion. *Marketing Sci.* 30(2): 195–212.

Jackson MO, Zenou Y (2014) Games on networks. Young P, Zamir S, eds. *Handbook of Game Theory*, vol. 4 (North Holland, Amsterdam), 95–163.

Katz M, Shapiro C (1985) Network externalities, competition and compatibility. *Amer. Econom. Rev.* 75(3):424–440.

Kempe D, Kleinberg J, Tardos É (2003) Maximizing the spread of influence through a social network. *Proc. 9th ACM SIGKDD Internat. Conf. Knowledge Discovery Data Mining* (ACM, New York), 137–146.

Lovett MJ, Peres R, Shachar R (2013) On brands and word of mouth. *J. Marketing Res.* 50(4):427–444.

Luo ZQ, Pang JS, Ralph D (1996) *Mathematical Programs with Equilibrium Constraints* (Cambridge University Press, Cambridge, UK).

Mahajan V, Muller E, Bass FM (1991) New product diffusion models in marketing: A review and directions for research. Nakicenovic N, Grübler A, eds. *Diffusion of Technologies and Social Behavior* (Springer, Berlin, Heidelberg), 125–177.

Nakkas A, Xu Y (2015) The impact of valuation heterogeneity and network structure on equilibrium prices in supply chain networks. Working paper, University of Texas at Arlington, Arlington.

Nemhauser GL, Wolsey LA (1988) *Integer and Combinatorial Optimization*, vol. 18 (Wiley, New York).

Schaefer Mark (2012) *Return on Influence: The Revolutionary Power of Klout, Social Scoring, and Influence Marketing* (McGraw-Hill, New York).

Seeman L, Singer Y (2013) Adaptive seeding in social networks. *Foundations Computer Science (FOCS), 2013 IEEE 54th Annual Sympos.* (IEEE, Piscataway, NJ), 459–468.

Stremersch S, Tellis GJ, Franses PH, Binken JL (2007) Indirect network effects in new product growth. *J. Marketing* 71(3):52–74.