Appendix A: Weighted bipartite graph for the example in Section 2.1

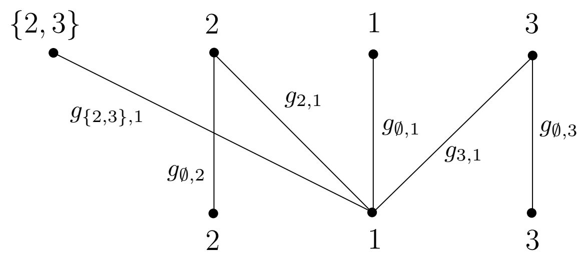

Figure 9 Example of the weighted bipartite graph.

Appendix B: Proof of Theorem 1

1. The existence of a PNE for any given price vector follows from the fact that the second stage game is of strategic complements. In other words, for any $\alpha_i \geq \alpha_i'$ and $\boldsymbol{\alpha_{-i}} \geq \boldsymbol{\alpha'_{-i}}$ (componentwise), we have increasing differences, that is, $u_i(\alpha_i, \boldsymbol{\alpha_{-i}}, p_i) - u_i(\alpha_i', \boldsymbol{\alpha_{-i}}, p_i) \geq u_i(\alpha_i, \boldsymbol{\alpha'_{-i}}, p_i) - u_i(\alpha_i', \boldsymbol{\alpha'_{-i}}, p_i)$. This follows from the positive externality assumption (see Assumption 1.b). Consequently, using the result from Theorem 1 in Section 3.2 of Jackson and Zenou (2014) (the same result can also be found in Topkis (1979)), we conclude the existence of a PNE.

2. If there are no ties in any of the PNEs, one can take $\epsilon = 0$. Consider the case where there are ties for some agents in one or more PNEs. In this case, one can choose $\epsilon > 0$ to be very small such that (i) agents that were buying in any PNE are still buying (in particular, their utility strictly increases and they become better-off), (ii) agents that were not buying (i.e., have negative utility) continue not to buy, and (iii) agents who were indifferent (i.e., exactly at zero utility), become strictly better-off as they derive a positive utility when the price is reduced by ϵ. Consequently, all the ties are eliminated for all PNEs.

3. In the case of a unique PNE, this is by definition a Pareto optimal PNE for the agents. We next consider a setting where there are multiple equilibria. Consider any agent whose actions differ in the different PNEs. As there are two actions, one of the actions is to buy in one of the PNEs. Note that when the agent buys, s/he derives a (strictly) positive utility with the perturbed prices and zero utility otherwise. As a result, this agent prefers to buy in the Pareto optimal solution. By using the non-negativity assumption, this agent can only non-negatively impact other agents' valuations and increase their utility. This implies that all other agents also prefer the buying decision of the focal agent. Similarly, one can argue that all agents who buy in one of the equilibria will buy (and prefer) the Pareto solution. The only agents who remain are the ones that do not buy in any equilibria. Note that they will not buy in the Pareto optimal solution either. This Pareto optimal solution is also a PNE: the buyers have no incentive to deviate as they derive a strictly positive utility and the non-buyers should not deviate either due to their negative utility from buying. $\square$

Appendix C: General Z-MIP formulation

In this section, we present the generalization of Z for any $\Gamma < K$. Recall that in Section 4 we presented the formulation for the case when $\Gamma = K-1$.

$$\max_{\substack{\mathbf{p}\in\mathbf{P}\\ \mathbf{y},\mathbf{z},\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\eta}}} \sum_{i\in\mathcal{I}} \left(z_i - c\alpha_i\right) \tag{Z-MIP}$$

s.t.

$$\left.\begin{aligned} y_i &= \sum_{\substack{|S|\le\Gamma\\ S\subset\mathcal{I}\setminus\{i\}}} g_{S,i}\alpha_{S\cup i} + \sum_{\substack{\Gamma<|S|<K\\ S\subset\mathcal{I}\setminus\{i\}}} g_{S,i}\eta_{S,i} - z_i \\ y_i &\ge \sum_{\substack{|S|\le\Gamma\\ S\subset\mathcal{I}\setminus\{i\}}} g_{S,i}\alpha_{S} + \sum_{\substack{\Gamma<|S|<K\\ S\subset\mathcal{I}\setminus\{i\}}} g_{S,i}\beta_{S} - p_i \\ y_i &\ge 0 \end{aligned}\right\} \quad \forall\, i\in\mathcal{I} \tag{C.1}$$

$$\left.\begin{aligned} z_i &\ge 0 \\ z_i &\le p_i \\ z_i &\le \alpha_i p^{max} \\ z_i &\ge p_i - (1-\alpha_i)p^{max} \end{aligned}\right\} \quad \forall\, i\in\mathcal{I} \tag{C.2}$$

$$\left.\begin{aligned} \alpha_S &\ge 0 \\ \alpha_S &\le \alpha_{S\setminus\{i\}} \qquad \forall\, i\in S \end{aligned}\right\} \quad \forall\, 1<|S|<\Gamma+2, S\subset\mathcal{I} \tag{C.3}$$

$$\alpha_{S\cup\{i,j\}} \ge \alpha_{S\cup\{i\}} + \alpha_{S\cup\{j\}} - \alpha_S \qquad \forall\, |S|<\Gamma, S\subset\mathcal{I}\setminus\{i,j\}, \{i\}\neq\{j\}\subset\mathcal{I} \tag{C.4}$$

$$\alpha_{S'} \le \beta_S \le 1 \qquad \forall\, |S'|=\Gamma, \Gamma<|S|<K, S'\subset S\subset\mathcal{I} \tag{C.5}$$

$$\beta_S \le \sum_{|S|'=\Gamma, S'\subset S} \alpha_{S'} \qquad \forall\, \Gamma<|S|<K, S\subset\mathcal{I} \tag{C.6}$$

$$\left.\begin{aligned} \eta_{S,i} &\ge 0 \\ \eta_{S,i} &\le \alpha_i \\ \eta_{S,i} &\le \beta_S \\ \eta_{S,i} &\ge \beta_S + \alpha_i - 1 \end{aligned}\right\} \quad \forall\, \Gamma<|S|<K, S\subset\mathcal{I}\setminus\{i\}, i\in\mathcal{I} \tag{C.7}$$

$$\alpha_i \in \{0,1\} \qquad \forall\, i\in\mathcal{I} \tag{C.8}$$

$$\alpha_\emptyset = 1 \tag{C.9}$$

The sets of constraints (C.3), (C.4), (C.5), and (C.6) linearize and ensure the correctness of the variables α_S and β_S. For example, constraint (C.4) for agents i and j and $S=\emptyset$ is given by: $\alpha_{i,j} \ge \alpha_i + \alpha_j - 1$, which along with other constraints ensures $\alpha_{i,j} = \alpha_i\alpha_j$. On the other hand, constraint (C.6) for $S=\{i,j\}$ and $\Gamma=1$ is given by: $\beta_{i,j} \le \alpha_i + \alpha_j$, which again along with other constraints ensures $\beta_{i,j} = \max\{\alpha_i, \alpha_j\}$. Finally, constraints (C.7) linearize the $\eta_{S,i}$ variable in a similar fashion as in constraints (C.3) and (C.4).

Appendix D: Proof of Theorem 2

Before proving the main theorem, we state and prove the next Lemma that identifies the optimal values of all variables, given α_i $\forall i \in \mathcal{I}$ (discrete or fractional).

Lemma 1. *For given (discrete or fractional) α_i $\forall i \in \mathcal{I}$, the revenue maximizing solution (and hence profit maximizing because α_i values are fixed) is given by:*

$$p_i^* = z_i^* + (1-\alpha_i)p^{\max} \qquad \forall i \in \mathcal{I} \tag{D.1}$$

$$z_i^* = \sum_{\substack{S\subset I\setminus\{i\}\\|S|\le\Gamma}} g_{S,i}\alpha^*_{S\cup i} + \sum_{\substack{S\subset I\setminus\{i\}\\|S|>\Gamma}} g_{S,i}\eta^*_{S,i} \qquad \forall i \in \mathcal{I} \tag{D.2}$$

$$\eta^*_{S,i} = \min\{\beta^*_S, \alpha_i\} \qquad \forall\ i \notin S, \Gamma < |S| < K, S \subset \mathcal{I}, i \in \mathcal{I} \tag{D.3}$$

$$\beta^*_S = \min\left\{1, \sum_{S'\subset S, |S'|=\Gamma} \alpha^*_{S'}\right\} \qquad \forall\ \Gamma < |S| < K, S \subset \mathcal{I}, \tag{D.4}$$

$$\alpha^*_S = \min_{i\in S}\{\alpha_i\} \qquad \forall\ 1 < |S| \le \Gamma+1, S \subset \mathcal{I}. \tag{D.5}$$

We begin by showing (D.1–D.2) first assuming that all remaining variables $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\eta}$ are given. Consider the remainder of the feasibility constraints (4.6) for each agent. Eliminating y_i reduces them to:

$$z_i \ge \max\{0, p_i - (1-\alpha_i)p^{\max}\} \tag{D.6}$$

$$z_i \le \min\Big\{p_i, \alpha_i p^{\max}, \sum_{\substack{S\subset I\setminus\{i\}\\|S|\le\Gamma}} g_{S,i}\alpha_{S\cup i} + \sum_{\substack{S\subset I\setminus\{i\}\\|S|>\Gamma}} g_{S,i}\eta_{S,i}, \sum_{\substack{S\subset I\setminus\{i\}\\|S|\le\Gamma}} g_{S,i}\left[\alpha_{S\cup i} - \alpha_S\right] + \sum_{\substack{S\subset I\setminus\{i\}\\|S|>\Gamma}} g_{S,i}\left[\eta_{S,i} - \beta_S\right] + p_i\Big\}. \tag{D.7}$$

We know that $g_{S,i}\alpha_{S\cup i} \le g_{S,i}\alpha_i$ and $g_{S,i}\eta_{S,i} \le g_{S,i}\alpha_i$. Therefore, $\sum_{\substack{S\subset I\setminus\{i\}\\|S|\le\Gamma}} g_{S,i}\alpha_{S\cup i} + \sum_{\substack{S\subset I\setminus\{i\}\\|S|>\Gamma}} g_{S,i}\eta_{S,i} \le \sum_{S\subset I\setminus\{i\}} g_{S,i}\alpha_i \le p^{\max}\alpha_i$, where the last inequality follows from the definition of $p^{\max}$. Note that the objective aims to maximize z_i. Since p_i is also a decision variable and increasing p_i increases the value for z_i, one can set $p_i^* = z_i^* + (1-\alpha_i)p^{\max}$. Therefore, we obtain:

$$0 \le z_i \le \sum_{\substack{S\subset I\setminus\{i\}\\|S|\le\Gamma}} g_{S,i}\alpha_{S\cup i} + \sum_{\substack{S\subset I\setminus\{i\}\\|S|>\Gamma}} g_{S,i}\eta_{S,i} \tag{D.8}$$

$$0 \le \sum_{\substack{S\subset I\setminus\{i\}\\|S|\le\Gamma}} g_{S,i}\left[\alpha_{S\cup i} - \alpha_S\right] + \sum_{\substack{S\subset I\setminus\{i\}\\|S|>\Gamma}} g_{S,i}\left[\eta_{S,i} - \beta_S\right] + (1-\alpha_i)p^{\max}. \tag{D.9}$$

We next show that constraint (D.9), which is independent of z_i, always holds allowing us to identify z^*. Observe that $[\alpha_S - \alpha_{S\cup i}] \le [1-\alpha_i]$. The inequality follows from constraint (C.4) when $\{j\} = \emptyset$. Similarly, $[\beta_S - \eta_{S,i}] \le (1-\alpha_i)$ using the last constraint of (C.7). These inequalities along with the non-negative influence terms and the definition of $p^{\max}$ prove (D.9). Therefore, maximizing z_i results in $z_i^* = \sum_{\substack{S\subset I\setminus\{i\}\\|S|\le\Gamma}} g_{S,i}\alpha_{S\cup i} + \sum_{\substack{S\subset I\setminus\{i\}\\|S|>\Gamma}} g_{S,i}\eta_{S,i}$, hence proving our claim about (D.2).

Given the form of z^* and the fact that we are maximizing it, α_S and $\eta_{S,i}$ should be set at their maximum values as the influence terms are always non-negative. Assuming that all $\boldsymbol{\alpha}$ values are given, (D.3) and (D.4) follow from constraints (C.7) and constraints (C.5–C.6) respectively. We next show that the solution in (D.5) is also the largest feasible solution for every α_S variable. One can see that for $|S|=2$ the result holds (similar to (D.3) above). We next show inductively that (D.5) holds for larger $|S|$ values. Assume that it is true for some $|S|=k$. Constraints (C.3–C.4) are as follows:

$$\begin{aligned}
&\alpha_S \leq \min_{k\in S\setminus\{i\}} \alpha_k && \forall\, i\in S\\
&\alpha_S \geq \min_{k\in S\setminus\{i\}} \alpha_k + \min_{k\in S\setminus\{j\}} \alpha_k - \min_{k\in S\setminus\{i,j\}} \alpha_k && \forall\, S\ni\{i\}, S\ni\{j\}, \{i\}\neq\{j\}\subset\mathcal{I}.
\end{aligned}$$

The largest feasible value of α_S from the first set of constraints is $\min_{i\in S}\left[\min_{k\in S\setminus\{i\}}\alpha_k\right] = \min_{k\in S}\alpha_k$. We next show that this is feasible to the second constraint. Observe that $\min_{k\in S}\alpha_k$ is the same as $\min_{k\in S\setminus\{i\}}\alpha_k$ or $\min_{k\in S\setminus\{j\}}\alpha_k$ (or both if they are equal). In addition, $\min_{k\in S\setminus\{i,j\}}\alpha_k$ is always greater than either of those terms and therefore, the constraint holds. Note that by setting all variables at their largest values, we arrived at a feasible and revenue maximizing solution. □

Consider solving the relaxed version of Z-MIP, where the binary constraints for each $\alpha_i\ \forall\, i\in\mathcal{I}$ are replaced by: $0\leq\alpha_i\leq 1$. Let $V^*=\left(\boldsymbol{\alpha}^*,\boldsymbol{\beta}^*,\boldsymbol{\eta}^*,\mathbf{p}^*,\mathbf{y}^*,\mathbf{z}^*\right)$ be a fractional optimal solution to the relaxed problem with a corresponding objective Π^*. We construct a new solution with all α_is being integer and show its feasibility to problem Z-MIP (hence relaxed Z-MIP as well) with an objective at least as good as V^*. We denote the solution we construct by $\tilde{V}=\left(\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}},\tilde{\boldsymbol{\eta}},\tilde{\mathbf{p}},\tilde{\mathbf{y}},\tilde{\mathbf{z}}\right)$ and its corresponding objective by $\tilde{\Pi}$. We next construct this new solution.

For any agent i, $\tilde{\alpha}_i=\lceil\alpha_i^*\rceil$ where $\lceil.\rceil$ refers to the ceiling function that maps a real number to the smallest following integer. Let T be the subset of agents who buy in $\tilde{V}$, i.e., $\tilde{\alpha}_i=1$ which also refers to those with $\alpha_i^*>0$. Then, $\tilde{\alpha}_S=1$ for any $S\subset T$ and $|S|\leq\Gamma+1$, and 0 otherwise. In addition, $\tilde{\beta}_S=1$ for all sets $S\subset\mathcal{I}$ such that $|S|>\Gamma$ and $|S\cap T|\geq\Gamma$. Also, $\tilde{\eta}_{S,i}=1$ if both $\tilde{\beta}_S=1$ and $\tilde{\alpha}_i=1$. Finally, if $\tilde{\alpha}_i=1$, $\tilde{z}_i=\tilde{p}_i$ (described below) and $\tilde{y}_i=0$. Otherwise, $\tilde{z}_i=\tilde{y}_i=0$ and $\tilde{p}_i=p^{\max}$.

From Lemma 1, we know that:

$$\Pi^*=\sum_{i\in T}\Big[\sum_{\substack{S\subset I\setminus\{i\}\\|S|\leq\Gamma}} g_{S,i}\alpha^*_{S\cup i}+\sum_{\substack{S\subset I\setminus\{i\}\\|S|>\Gamma}} g_{S,i}\eta^*_{S,i}\Big]-c\sum_{i\in T}\alpha_i^*=\sum_{i\in T}\Big[\sum_{\substack{S\subset T\setminus\{i\}\\|S|\leq\Gamma}} g_{S,i}\alpha^*_{S\cup i}+\sum_{\substack{S\subset I\setminus\{i\}\\|S|>\Gamma,|S\cap T|\geq\Gamma}} g_{S,i}\eta^*_{S,i}\Big]-c\sum_{i\in T}\alpha_i^*$$

$$\tilde{\Pi}=\sum_{i\in T}\Big[\sum_{\substack{S\subset T\setminus\{i\}\\|S|\leq\Gamma}} g_{S,i}+\sum_{\substack{S\subset I\setminus\{i\}\\|S|>\Gamma,|S\cap T|\geq\Gamma}} g_{S,i}\Big]-c|T|.$$

We next reorder the agents, rewrite the objective function, and argue that a certain property holds for a partial sum. We then introduce a sequence of iterative steps to show that $\tilde{\Pi}\geq\Pi^*$.

- Order the agents in the set $T=\{k_1,k_2,\ldots,k_{|T|}\}$ such that $\alpha^*_{k_1}\geq\alpha^*_{k_2}\geq\cdots\geq\alpha^*_{k_{|T|}}$.
- Create the nested sets of agents: $T_1\subset T_2\subset\cdots\subset T_{|T|}=T$, where $\{k_1\}=T_1, T_1\cup\{k_2\}=T_2,\ldots,T_m\cup\{k_{m+1}\}=T_{m+1},\ldots,T_{|T|}=T$.

- Rewrite Π^* as follows:

$$\Pi^* = \sum_{m=1}^{|T|} \Big[\sum_{i \in T_{m-1}} \Big(\sum_{\substack{S \ni k_m \\ S \subset T_m \setminus \{i\}, |S| \leq \Gamma}} g_{S,i} \alpha^*_{S \cup i} + \sum_{\substack{S \ni k_m, S \subset I \setminus \{i\} \\ |S| > \Gamma, |S \cap T_m| = \Gamma}} g_{S,i} \eta^*_{S,i} \Big) + \sum_{\substack{S \subset T_{m-1} \\ |S| \leq \Gamma}} g_{S,k_m} \alpha^*_{S \cup k_m} + \sum_{\substack{S \subset I \setminus \{k_m\} \\ |S| > \Gamma, |S \cap T_m| \geq \Gamma}} g_{S,k_m} \eta^*_{S,k_m} - c\alpha^*_{k_m} \Big].$$

  We build the above objective by considering the marginal terms obtained by adding one agent at a time starting from k_1 to $k_{|T|}$. For every agent that we add, say k_m, related value terms and cost terms are included. The value term corresponds to all influence terms related to all sets S that consists of k_m. In particular, they consist of terms where k_m is the influencer for all agents added so far (i.e., $i \neq k_m, i \in T_{m-1}$) and terms where k_m is influenced by previous agents (i.e., by agents in T_{m-1} on $i = k_m$).
- Substituting (D.3–D.5) from Lemma 1 in Π^*, we obtain:

$$\Pi^* = \sum_{m=1}^{|T|} \Big[\sum_{i \in T_{m-1}} \Big(\sum_{\substack{S \ni k_m \\ S \subset T_m \setminus \{i\}, |S| \leq \Gamma}} g_{S,i} + \sum_{\substack{S \ni k_m, S \subset I \setminus \{i\} \\ |S| > \Gamma, |S \cap T_m| = \Gamma}} g_{S,i} \Big) + \sum_{\substack{S \subset T_{m-1} \\ |S| \leq \Gamma}} g_{S,k_m} + \sum_{\substack{S \subset I \setminus \{k_m\} \\ |S| > \Gamma, |S \cap T_m| \geq \Gamma}} g_{S,k_m} - c \Big] \alpha^*_{k_m}. \tag{D.10}$$

We know that $\Pi^* \geq 0$ because a no-buy for everyone results in 0 profit and is a feasible solution. Moreover, any cumulative sum starting from the last term in (D.10) has to be non-negative:

$$\Pi^*_{[l,|T|]} \geq 0 \quad \forall l = 1, \ldots, |T|. \tag{D.11}$$

Otherwise, it would have been beneficial to set α_{k_m} values corresponding to these agents $(\alpha_{k_l}, \ldots \alpha_{k_{|T|}})$ to 0, and restrict T to just T_{l-1} (where $T_0 = \emptyset$).

We now introduce a sequence of iterative steps to show that $\tilde{\Pi} \geq \Pi^*$. We choose a decreasing iteration counter, l, and show that the above property holds in each step.

1. Let $l = |T|$, $V^l = V^*$ and $\Pi^l = \Pi^*$.
2. If $\exists\ i \in \mathcal{I}$ s.t. $0 < \alpha^l_i < 1$, go to step 3. Otherwise, set $\tilde{\Pi} = \Pi^l$ and the procedure terminates.
3. For all $i \in \{k_l, \ldots, k_{|T|}\}$, increase $\alpha^{l-1}_i = \alpha^l_{k_{l-1}}$ and no change otherwise, i.e., $\alpha^{l-1}_i = \alpha^l_i$.

   This results in $\Pi^{l-1}_{[l,|T|]} \geq \Pi^l_{[l,|T|]}$ as $\Pi^l_{[l,|T|]} \geq 0$ and $\Pi^{l-1}_{[i,i]} = \Pi^l_{[i,i]}\ \forall i = 1, \ldots, l-1$. Therefore, we obtain $\Pi^{l-1}_{[1,|T|]} \geq \Pi^l_{[1,|T|]}$. Also, from the previous step (or (D.11) when $l = |T|$), we know that $\Pi^l_{[i,|T|]} \geq 0\ \forall\ i = 1, \ldots, |T|$. Therefore, $\Pi^{l-1}_{[i,|T|]} \geq 0\ \forall\ i = 1, \ldots, |T|$.
4. Proceed back to step 2 after setting $l := l - 1$.

As a result, the algorithm terminates in at most $|T|$ steps and the final solution is such that all α_i values are integer as $\alpha_{k_0} = 1$. Note that in the above steps, we did not discuss the feasibility of the solution in each step. It can be shown that the solution remains feasible to the relaxed Z-MIP in each iteration, although it is not relevant from the perspective of the proof and hence omitted. The final integer solution $\tilde{\alpha}$ is feasible to the relaxed Z-MIP and also to Z-MIP with an objective $\tilde{\Pi} \geq \Pi^*$, hence concluding the proof. $\square$

Appendix E: Proof of correctness of Algorithm 1

First, we note that after each iteration of the procedure, at least one agent is removed from the network. Therefore, the algorithm clearly terminates in finite time, more precisely, at most after N iterations. We denote by $I_T(\leq N)$ the total number of iterations and by $N^{(t)}$ the number of agents in the network at iteration $t = 1, 2, \dots, I_T$.

Next, we show that the only candidates for the optimal uniform price are $p_{min}^{(t)}\ \forall\ t \in \{1, \dots, I_T\}$ (see the definition in Algorithm 1). First, observe that the uniform optimal price cannot be smaller than $p_{min}^{(1)}$. Indeed, for any price $p \leq p_{min}^{(1)}$, all agents that bought in the discriminative case will still buy at this smaller price. However, a lower price than $p_{min}^{(1)}$ will result in lower profit (per buyer) for the seller. It is possible though that some new agents would buy the item at the lower price inducing an overall higher profit. Nevertheless, one can see that the new lower price in a uniform pricing scheme certainly will not be less than c. Therefore, it suffices to consider prices that are at least c but lower than $p_{min}^{(1)}$, if any. If this is the case, it would have been profitable to offer this price (which is higher than c) to those agents in the discriminative pricing scheme as well. Since it was not optimal to offer a lower price than $p_{min}^{(1)}$ to the non-buyers, it is not profitable to decrease the uniform price lower than $p_{min}^{(1)}$. As a result, we conclude that the optimal uniform price cannot be smaller than $p_{min}^{(1)}$. We now consider the case where the uniform price is larger than $p_{min}^{(1)}$. In this case, we lose the buyers with $p_i \leq p_{min}^{(1)}$ from the discriminative pricing scheme. Otherwise, in the discriminative case one would offer a higher price. We can therefore remove those agents from the network. Now applying the same argument, it is the case that the uniform optimal price cannot be equal to a value that is strictly between $p_{min}^{(1)}$ and $p_{min}^{(2)}$. By repeating this procedure, we conclude that the optimal uniform price has to be equal to one of the $p_{min}^{(t)}$ prices. In order to select the best uniform price among these I_T candidates, we just need to evaluate the corresponding profits (denoted by $\Pi^{(t)}\ \forall t = 1, 2, \dots, I_T$) and choose the one that yields the maximal profit. One can do so by using the following relation:

$$\Pi^{(t)} = \left(p_{min}^{(t)} - c\right) \sum_{i=1}^{N^{(t)}} \alpha_i^{(t)}, \tag{E.1}$$

where $N^{(t)}$ is the remaining number of agents in the network at iteration t. □

Appendix F: Proof of Proposition 2

Consider the continuous relaxation of problem Zi that replaces the binary constraint $\alpha_i \in \{0, 1\}$ by $0 \leq \alpha_i \leq 1\ \forall\ i \in \mathcal{I}$. We consider the version of problem Zi without the dual variables w_i (see Observation 4). Let $V^* = \left(p_i^*, d_i^*, y_i^*, \alpha_i^*, \gamma_i^*\right)\ \forall i \in \mathcal{I}$ be an optimal solution for the relaxed problem with the corresponding objective Π^*. We divide the proof into two parts. First, we show that given any optimal solution, one can construct a new optimal solution for which all variables $\alpha_i^*\ \forall i \in \mathcal{I}$ are integer. Second, we construct from the latter solution a new solution with all variables $\gamma_i^*\ \forall i \in \mathcal{I}$ integer as well. Assume that the initial optimal solution has at least one fractional component i.e.,

$\exists\ j \in \mathcal{I}$ s.t. $0 < \alpha_j^* < 1$. Now, consider three other feasible solutions $\tilde{V}, \overline{V}$, and $\underline{V}$ to the relaxed problem as follows:

$$\tilde{V} = \left(\tilde{p}_i, \tilde{d}_i, \tilde{y}_i, \tilde{\alpha}_i, \tilde{\gamma}_i\right) = \begin{cases} \left(p_i^*, d_i^*, y_i^*, 1, \gamma_i^*\right) & \text{if } i = j \\ V_i^* & \forall i \in S_j \\ V_i^* & \text{otherwise} \end{cases}$$

$$\overline{V} = \left(\overline{p}_i, \overline{d}_i, \overline{y}_i, \overline{\alpha}_i, \overline{\gamma}_i\right) = \begin{cases} \left(p_i^*, d_i^*, 0, 1, 1\right) & \text{if } i = j \\ \left(p_i^* + (1-\gamma_j^*) g_{ji}, d_i^*, y_i^*, \alpha_i^*, \gamma_i^*\right) & \forall i \in S_j \\ V_i^* & \text{otherwise} \end{cases}$$

$$\underline{V} = \left(\underline{p}_i, \underline{d}_i, \underline{y}_i, \underline{\alpha}_i, \underline{\gamma}_i\right) = \begin{cases} \left(p_i^*, d_i^*, 0, 1, 0\right) & \text{if } i = j \\ \left(p_i^* - \gamma_j^* g_{ji}, d_i^*, y_i^*, \alpha_i^*, \gamma_i^*\right) & \forall i \in S_j \\ V_i^* & \text{otherwise} \end{cases}$$

Here, S_j denotes the set of neighbors of agent j (excluding j). We observe that all three solutions are feasible to the problem for the following reasons. First, since $0 < \alpha_j^* < 1$ it implies that $\big(g_{jj} + \sum_i \gamma_i^* g_{ij} - p_j^*\big) = 0$ as otherwise it cannot be a best response for agent j and cannot satisfy the equilibrium constraints. In addition, to ensure feasibility, we should have either $\gamma_j^* = 0$ or $d_j^* = t_j$ and hence, $y_j^* = 0$. Therefore, changing α_j^* to 1 or 0 does not affect the best response of agent j as far as α_j is concerned. Note that we construct the dual variable for agent j to satisfy all feasibility constraints. Second, we have modified the prices of the neighbors of agent j exactly by the change in the level of influence from agent j on them and therefore, purchasing decisions of agents $i \in S_j$ remain the same. Third, since agents in $\mathcal{I} \backslash (\{j\} \cup S_j)$ are unaffected by the change in α_j^* or p_i^* $\forall i \in S_j$, the solution remains feasible for them as well.

We denote the objective corresponding to these new solutions by $\tilde{\Pi}$, $\overline{\Pi}$, and $\underline{\Pi}$ respectively. We observe that $\Pi^* - \tilde{\Pi} = -(1-\alpha_j^*)(p_j^* - c)$. Since V^* is an optimal solution, it has to be the case that $p_j^* - c \leq 0$ as otherwise $\tilde{V}$ is a better solution. In addition, we observe that $\Pi^* - \overline{\Pi} = -(1-\alpha_j^*)(p_j^* - c) + (1-\gamma_j^*) d_j^* - \sum_{i \in S_j} \alpha_i^* g_{ji} (1-\gamma_j^*)$ and $\Pi^* - \underline{\Pi} = \alpha_j^* (p_j^* - c) - \gamma_j^* d_j^* + \sum_{i \in S_j} \alpha_i^* g_{ji} \gamma_j^*$. Since Π^* is the optimal value of the objective and $0 < \alpha_j^* < 1$, both $\overline{\Pi}$ and $\underline{\Pi}$ are lower or equal than Π^*. By requiring $\Pi^* - \overline{\Pi} \geq 0$ together with $\Pi^* - \underline{\Pi} \geq 0$ and using the fact that $p_j^* - c \leq 0$, we obtain the condition: $\alpha_j \geq \gamma_j$. However, from feasibility, we know that $\alpha_j \leq \gamma_j$ and thus $\alpha_j = \gamma_j$. By using this fact, we obtain: $\Pi^* - \overline{\Pi} = -(1-\alpha_j^*)\big(p_j^* - c - d_j^* + \sum_{i \in S_j} \alpha_i^* g_{ji}\big)$ and $\Pi^* - \underline{\Pi} = \alpha_j^* \big(p_j^* - c - d_j^* + \sum_{i \in S_j} \alpha_i^* g_{ji}\big)$. Since $0 < \alpha_j^* < 1$, it has to be the case that both $\overline{V}$ and $\underline{V}$ are also optimal solutions as they are feasible and yield the same objective as V^*. We therefore have reduced the number of fractional components by one. One can now repeat the same procedure for each fractional value α_j^* to derive a constructive way of identifying a feasible integral solution with an objective as good as the initial fractional solution. Since the number of agents is finite, this step is repeated at most N times.

Then, we conclude that the continuous relaxation of problem Zi is tight, meaning that for any feasible fractional solution, one can find an integral solution with at least the same objective.

At this point, we know there exists an optimal solution with α_i^* integer $\forall i \in \mathcal{I}$. We next show the following result that allows to guarantee the integrality of γ_i^* $\forall i \in \mathcal{I}$ at optimality. In other words, it is optimal for each buyer to either fully influence (i.e., $\alpha_i^* = \gamma_i^* = 1$) and receive the full discount or not influence at all (i.e., $\gamma_i^* = 0$) and pay the full price. Consider the optimal integer purchasing decisions α_i^* $\forall i \in \mathcal{I}$. For all agents k with $\alpha_k^* = 0$, it is clear from feasibility that $\gamma_k^* = 0$. Consider a given optimal solution denoted by V^* with $\alpha_j^* = 1$ and assume by contradiction that $0 < \gamma_j^* < 1$. Consider the following feasible solutions (denoted by $\overline{V}$ and $\underline{V}$) to the relaxed problem Zi:

$$\overline{V} = \left(\overline{p}_i, \overline{d}_i, \overline{y}_i, \overline{\alpha}_i, \overline{\gamma}_i\right) = \begin{cases} \left(p_i^*, d_i^*, y_i^*, 1, 1\right) & \text{if } i = j \\ \left(p_i^* + (1-\gamma_j^*)g_{ji}, d_i^*, y_i^*, \alpha_i^*, \gamma_i^*\right) & \forall i \in S_j \\ V_i^* & \text{otherwise} \end{cases}$$

$$\underline{V} = \left(\underline{p}_i, \underline{d}_i, \underline{y}_i, \underline{\alpha}_i, \underline{\gamma}_i\right) = \begin{cases} \left(p_i^*, d_i^*, y_i^*, 1, 0\right) & \text{if } i = j \\ \left(p_i^* - \gamma_j^* g_{ji}, d_i^*, y_i^*, \alpha_i^*, \gamma_i^*\right) & \forall i \in S_j \\ V_i^* & \text{otherwise} \end{cases}$$

As before, S_j denotes the set of neighbors of agent j (excluding j). We observe that both solutions are feasible to the problem for the following reasons. First, we note that we construct the dual variables for agent j to satisfy all feasibility constraints. Indeed, since $0 < \gamma_j^* < 1$, it has to be the case that $d_j^* = t_j$ as otherwise it cannot be a best response for agent j and cannot satisfy the equilibrium constraints. Second, we have modified the prices of neighbors of agent j exactly by the change in the level of influence from agent j on them and hence, the purchasing decisions of agents $i \in S_j$ remain the same. Third, since agents in $\mathcal{I} \backslash (\{j\} \cup S_j)$ are unaffected by the change in α_j^* or p_i^* $\forall i \in S_j$, the solution also remains feasible for them.

We denote the objective corresponding to these new solutions by $\overline{\Pi}$ and $\underline{\Pi}$ respectively. We observe that $\Pi^* - \overline{\Pi} = (1-\gamma_j^*)\left[d_j^* - \sum_{i \in S_j} \alpha_i^* g_{ji}\right]$ and $\Pi^* - \underline{\Pi} = -\gamma_j^* \left[d_j^* - \sum_{i \in S_j} \alpha_i^* g_{ji}\right]$. Since Π^* is the optimal value of the objective and $0 < \gamma_j^* < 1$, $d_j^* - \sum_{i \in S_j} \alpha_i^* g_{ji} = 0$ should hold. Otherwise, one of the solutions we constructed is strictly better than the optimal solution and this is a contradiction. Consequently, one can see that both $\overline{V}$ and $\underline{V}$ are also optimal solutions as they are feasible and yield the same objective as V^*. In the process, we have therefore reduced the number of fractional components by one. One can now repeat the same procedure for each fractional value γ_j^* to derive a constructive way of identifying a feasible integral solution to the problem with an objective as good as the fractional solution. Note that since the number of agents is finite, this step is repeated at most N times. In conclusion, the continuous relaxation of problem Zi always has an optimal solution with integer purchasing decisions and integer γ variables. $\square$

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