EC.1. Proof of Lemma 1

Proof. 1. Since the proof may not be easy to follow, we present it together with a concrete example to illustrate the different steps. Let $T=6,\ q^0=7,\ A=\{(1,1),(3,3)\},\ B=\{(3,3)\}$ and $(t',k')=(5,5)$. We denote by $POP_t(\mathbf{p}_A)$ the profits at time t for the price vector $\mathbf{p}_A$. In addition, we further assume that: $\delta_5=g_4(1)=0.8,\ \delta_6=g_5(1)=0.9$. We next define the following quantities:

$$\begin{aligned} a_t &= POP_t(\mathbf{p}_A) &&= POP_t(1,7,3,7,7,7)\\ a'_t &= POP_t(\mathbf{p}_{A\cup(t',k')}) &&= POP_t(1,7,3,7,5,7)\\ b_t &= POP_t(\mathbf{p}_B) &&= POP_t(7,7,3,7,7,7)\\ b'_t &= POP_t(\mathbf{p}_{B\cup(t',k')}) &&= POP_t(7,7,3,7,5,7). \end{aligned}$$

For each time t, we define the following coefficient:

$$\delta_t=\frac{g_1((\mathbf{p}_A)_{t-1})\cdot g_2((\mathbf{p}_A)_{t-2})\cdots g_{t-1}((\mathbf{p}_A)_1)}{g_1((\mathbf{p}_B)_{t-1})\cdot g_2((\mathbf{p}_B)_{t-2})\cdots g_{t-1}((\mathbf{p}_B)_1)}.$$

δ_t represents the multiplicative reduction in demand at time t from the promotions present in the set A but not in B. Observe that from Assumption 4, we have $o\le\delta_{t'}\le\delta_{t'+1}\le\cdots\le\delta_T\le 1$. In addition, we have: $a_t=\delta_t b_t,\ a'_t=\delta_t b'_t$. Observe also that condition (14) is equivalent to:

$$\sum_{t=1}^{T}a'_t-\sum_{t=1}^{T}a_t\ge 0. \tag{EC.1}$$

Note that $a_t=a'_t$ for all $t<t'$. In the example, we have $a_1=a'_1,\dots,a_4=a'_4$ as the prices in periods 1-4 are the same. Therefore, (EC.1) becomes: $\sum_{t=t'}^{T}a'_t\ge\sum_{t=t'}^{T}a_t$. In the example, we obtain: $a'_5+a'_6\ge a_5+a_6$. Note that $a'_t\le a_t$ for any $t>t'$. In the example, a'_t has a promotion at $t=5$. However, there is no promotion in a_t at $t=5$ and therefore, the objective at $t=6$ for a'_t is lower than the one in a_t, i.e., $a'_6\le a_6$. This implies that:

$$a'_{t'}-a_{t'}\ge\sum_{t=t'+1}^{T}(a_t-a'_t)\ge 0.$$

In the example, this translates to $a'_5-a_5\ge a_6-a'_6\ge 0$. We next multiply the left hand side by $1/\delta_{t'}$ and the terms in the right hand side by $1/\delta_t$ (recall that $1/\delta_{t'}\ge 1/\delta_t$ for $t>t'$). Therefore, we obtain:

$$b'_{t'}-b_{t'}=\frac{a'_{t'}-a_{t'}}{\delta_{t'}}\ge\sum_{t=t'+1}^{T}\left(\frac{a_t-a'_t}{\delta_t}\right)=\sum_{t=t'+1}^{T}\left(b_t-b'_t\right)\ge 0.$$

In the example, this translates to: $b_5' - b_5 = \frac{a_5' - a_5}{0.8} \geq \frac{a_6 - a_6'}{0.9} = b_6 - b_6' \geq 0$. Recall that our goal is to show equation (15), or alternatively: $\sum_{t=1}^T a_t' - \sum_{t=1}^T a_t \leq \sum_{t=1}^T b_t' - \sum_{t=1}^T b_t$. Note that this is equivalent to: $\sum_{t=t'}^T (a_t' - a_t) = \sum_{t=t'}^T \delta_t (b_t' - b_t) \leq \sum_{t=t'}^T (b_t' - b_t)$. By rearranging the terms, we obtain:

$$\sum_{t=t'+1}^{T} (1-\delta_t)(b_t - b_t') \leq (1-\delta_{t'})(b_{t'}' - b_{t'}).$$

In the example, this would be: $0.1(b_6 - b_6') \leq 0.2(b_5' - b_5)$. Finally, note that the above inequality is true because of the following:

$$\sum_{t=t'+1}^{T} (1-\delta_t)(b_t - b_t') \leq \sum_{t=t'+1}^{T} (1-\delta_{t'})(b_t - b_t') \leq (1-\delta_{t'})(b_{t'}' - b_{t'}).$$

In the example, this is clear because: $0.1(b_6 - b_6') \leq 0.2(b_6 - b_6') \leq 0.2(b_5' - b_5)$. □

2. We first introduce the following notation. Let $\boldsymbol{\gamma}^{POP}$ be an optimal solution to the POP and $\{(t_1, k_1), \ldots, (t_n, k_n)\}$ the set of promotions in $\boldsymbol{\gamma}^{POP}$. For any subset $B \subset \{1, 2, \ldots, n\}$, we define: $\boldsymbol{\gamma}(B) = \boldsymbol{\gamma}(\{(t_i, k_i) : i \in B\})$. For example, let the price ladder be $\{q^0 = 5, q^1 = 4\}$ and $\boldsymbol{\gamma}^{POP} = \boldsymbol{\gamma}(\{(1,1), (3,1), (5,1)\})$. Then, $\boldsymbol{\gamma}(\{1,3\}) = \boldsymbol{\gamma}(\{(1,1), (5,1)\})$.

   Note that one can write the following telescoping sum:

$$POP(\boldsymbol{\gamma}^{POP}) = POP(\boldsymbol{\gamma}\{1\}) + \sum_{m=1}^{n-1} \Big[POP(\boldsymbol{\gamma}\{1, \ldots, m+1\}) - POP(\boldsymbol{\gamma}\{1, \ldots, m\}) \Big].$$

   Based on Proposition EC.1 below, we have for each $m = 1, 2, \ldots n-1$: $POP(\boldsymbol{\gamma}\{1, \ldots, m+1\}) - POP(\boldsymbol{\gamma}\{1, \ldots, m\}) \geq 0$. By applying the submodularity property from Lemma 1 part 1, we obtain: $0 \leq POP(\boldsymbol{\gamma}\{1, \ldots, m+1\}) - POP(\boldsymbol{\gamma}\{1, \ldots, m\}) \leq POP(\boldsymbol{\gamma}\{m+1\}) - POP(\boldsymbol{\gamma}^0)$. Therefore, we have:

$$\begin{aligned} POP(\boldsymbol{\gamma}^{POP}) &= POP(\boldsymbol{\gamma}\{1\}) + \sum_{m=1}^{n-1} \Big[POP(\boldsymbol{\gamma}\{1, \ldots, m+1\}) - POP(\boldsymbol{\gamma}\{1, \ldots, m\}) \Big] \\ &\leq POP(\boldsymbol{\gamma}^0) + \sum_{m=1}^{n} \Big[POP(\boldsymbol{\gamma}\{m\}) - POP(\boldsymbol{\gamma}^0) \Big] = LP(\boldsymbol{\gamma}^{POP}). \end{aligned}$$

Proposition EC.1. *Let $n \geq 2$ be an integer and $\boldsymbol{\gamma}^{POP}$ an optimal solution to the POP with n promotions. Then, $POP(\boldsymbol{\gamma}\{1, \ldots, m+1\}) - POP(\boldsymbol{\gamma}\{1, \ldots, m\}) \geq 0$ for $m = 1, 2, \ldots, n-1$.*

Proof. The proof proceeds by induction on the number of promotions. We first show that the claim is true for the base case i.e., $n = 2$. By the optimality of $\boldsymbol{\gamma}^{POP} = \boldsymbol{\gamma}\{1, 2\}$, we have:

$$0 \leq POP(\boldsymbol{\gamma}\{1, 2\}) - POP(\boldsymbol{\gamma}\{1, 2\}).$$

Next, we assume that the claim is true for n and show its correctness for $n+1$. Let POP' denote the POP problem with the additional constraint that promotion (t_1, k_1) is used, i.e., $p_{t_1} = q^{k_1}$. One can see that the set of promotions $\{(t_2, k_2), \dots, (t_{n+1}, k_{n+1})\}$ is an optimal solution to POP' with n promotions. Therefore, by using the induction hypothesis, we have:

$$\begin{array}{lll} POP'(\gamma\{2,\dots,n,n+1\}) & -POP'(\gamma\{2,\dots,n\}) & \geq 0 \\ \vdots & \vdots & \vdots \\ POP'(\gamma\{2,3\}) & -POP'(\gamma\{2\}) & \geq 0 \end{array}$$

Equivalently, in terms of the POP:

$$\begin{array}{lll} POP(\gamma\{1,\dots,n,n+1\}) & -POP(\gamma\{1,\dots,n\}) & \geq 0 \\ \vdots & \vdots & \vdots \\ POP(\gamma\{1,2,3\}) & -POP(\gamma\{1,2\}) & \geq 0 \end{array}$$

Therefore, it remains to show that: $POP(\gamma\{1,2\}) - POP(\gamma\{1\}) \geq 0$. We next prove the following chain of inequalities:

$$\begin{aligned} POP(\gamma\{1,2\}) - POP(\gamma\{1\}) &\geq POP(\gamma\{1,2,3\}) - POP(\gamma\{1,3\}) \\ &\geq POP(\gamma\{1,2,3,4\}) - POP(\gamma\{1,3,4\}) \\ &\vdots \\ &\geq POP(\gamma\{1,\dots,n,n+1\}) - POP(\gamma\{1,3,4,\dots,n,n+1\}). \end{aligned} \tag{EC.2}$$

By using the induction hypothesis together with the submodularity property from Lemma 1 part 1, we obtain for each $m = 2, 3, \dots, n-1$:

$$POP(\gamma\{1,\dots,m,m+1\}) - POP(\gamma\{1,\dots,m\}) \leq \\ POP(\gamma\{1,3,4,\dots,m,m+1\}) - POP(\gamma\{1,3,4,\dots,m\}).$$

Finally, from the optimality of $\gamma\{1,\dots,n,n+1\}$ for the POP, we have: $POP(\gamma\{1,\dots,n,n+1\}) - POP(\gamma\{1,\dots,n\}) \geq 0$. By rearranging the terms in the above equations, one can derive the chain of inequalities in (EC.2) and this concludes the proof. □

EC.2. Proof of Proposition 3

Proof. We denote the set of promotions in the price vector $\mathbf{p}$ by: $\mathbf{p} = \mathbf{p}\{(t_1, q^{t_1}), \dots, (t_N, q^{t_N})\}$, where N is the number of promotions. The price vector $\mathbf{p}^n = \mathbf{p}\{(t_n, q^{t_n})\}$ for each $n = 1, \dots, N$ denotes the single promotion price at time t_n (no promotion at the remaining periods). By convention, let us denote $n = 0$ to be the regular price only vector $\mathbf{p}^0 = (q^0, \dots, q^0)$. We denote the cumulative POP objective in periods $[u, v)$ when using $\mathbf{p}^n$ by:

$$x^n_{[u,v)} = POP(\mathbf{p}\{(t_n, q^{t_n})\})_{[u,v)} = \sum_{t=u}^{v-1} \mathbf{p}_t\{(t_n, q^{t_n})\} d_t(\mathbf{p}_t\{(t_n, q^{t_n})\}).$$

Note that he LP objective can be written as: $LP(\mathbf{p}) = x^0_{[1,T]} + \sum_{n=1}^{N}\left(x^n_{[1,T]} - x^0_{[1,T]}\right)$.

Since $\mathbf{p}^n$ and $\mathbf{p}^0$ do not promote before t_n, we have $x^n_{[1,t_n)} = x^0_{[1,t_n)}$. In addition, since $\mathbf{p}^n$ promotes at $t = t_n$ and $\mathbf{p}^0$ does not, the vector $\mathbf{p}^n$ yields a lower objective for the periods after t_n, i.e., $x^n_{[t_{n+1},T]} \leq x^0_{[t_{n+1},T]}$. Therefore, we obtain for each $n = 1,\ldots,N$:

$$x^n_{[1,T]} - x^0_{[1,T]} = x^n_{[1,t_n)} + x^n_{[t_n,t_{n+1})} + x^n_{[t_{n+1},T]} - x^0_{[1,t_n)} - x^0_{[t_n,t_{n+1})} - x^0_{[t_{n+1},T]} \leq x^n_{[t_n,t_{n+1})} - x^0_{[t_n,t_{n+1})}.$$

Therefore: $LP(\mathbf{p}) \leq UB = x^0_{[1,T]} + \sum_{n=1}^{N}\left(x^n_{[t_n,t_{n+1})} - x^0_{[t_n,t_{n+1})}\right) = x^0_{[1,t_1)} + \sum_{n=1}^{N} x^n_{[t_n,t_{n+1})}$.
Let UB_t denote the value of UB at time t. Specifically, if $t \in [t_n, t_{n+1})$, then $UB_t = x^n_t$. We can write for any feasible price vector $\mathbf{p}$: $POP(\mathbf{p}) = \sum_{t=1}^{T} a_t UB_t$, where a_t is the decrease in demand at time t due to the past promotions in $\mathbf{p}$. In particular, if $t_n < t \leq t_{n+1}$, then: $a_t = g_{t-t_1}(q^{t_1})g_{t-t_2}(q^{t_2})\cdots g_{t-t_n}(q^{t_n})$. Since $0 \leq \underline{R} \leq a_t \leq 1$, we obtain: $\underline{R}\cdot LP(\mathbf{p}) \leq \underline{R}\cdot UB \leq POP(\mathbf{p})$. □

EC.3. Proofs of Tightness for Multiplicative Demand

1. **Lower bound**

   Proof. In the case when $S \geq M$, we know from Proposition 1 that the LP approximation is exact. Therefore, the result holds in this case.

   We next consider that $S < M$ and construct an instance of the POP as well as a price vector $\boldsymbol{p}^*$. We then show that this price vector $\boldsymbol{p}^*$ is optimal for both the POP and the LP approximation.

   Let $T = L(M+1)$ and let us define the following price vector:

$$\boldsymbol{p}^* = \Big(q^K, \underbrace{q^0,\ldots,q^0}_{M\text{ times}}, q^K, \underbrace{q^0,\ldots,q^0}_{M\text{ times}}, \ldots, q^K, \underbrace{q^0,\ldots,q^0}_{M\text{ times}}\Big).$$

   Let $\mathcal{U} = \{1, (M+1)+1, 2(M+1)+1, \ldots, (L-1)(M+1)+1\}$ denote the set of promotion periods in $\boldsymbol{p}^*$. We choose the demand functions f_t to be:

$$f_t(p_t) = \begin{cases} Z/q^K & \text{if } t \in \mathcal{U} \text{ and } p_t = q^K, \\ 1/q^0 & \text{otherwise,} \end{cases}$$

   where:

$$Y = 1 + \sum_{m=1}^{M}(1 - g_m(q^K)),$$
$$Z = (M+2)Y.$$

   We define all the costs to be zero, i.e., $c_t = 0, \forall t = 1,\ldots,T$. We prove the proposition by the following steps:

Step 1: We show that $\boldsymbol{p}^*$ is an optimal LP solution.

Step 2: We show that there exists an optimal POP solution with promotions only during periods $t \in \mathcal{U}$.

Step 3: We show that if $\boldsymbol{p}$ promotes only during periods $t \in \mathcal{U}$, then $POP(\boldsymbol{p}) \leq POP(\boldsymbol{p}^*)$.

By combining steps 2 and 3, we conclude that $\boldsymbol{p}^*$ is an optimal POP solution. Consequently, $POP(\boldsymbol{p}^{POP}) = POP(\boldsymbol{p}^{LP})$, implying that the lower bound is tight.

Proof of Step 1. By definition, we have: $POP(\boldsymbol{p}\{(t,K)\}) = POP(\boldsymbol{p}^0) + Z - Y$ for $t \in \mathcal{U}$. Therefore the LP coefficients as defined in (6) are given by:

$$b_t^k = \begin{cases} Z - Y & \text{if } t \in \mathcal{U}, k = K, \\ \leq 0 & \text{otherwise.} \end{cases}$$

Any LP optimal solution selects at most L of γ_t^k, for $k = 1, \ldots, K$ to be 1. Consequently, the optimal LP objective is bounded above by $T + L(Z - Y)$. In fact, the following $\boldsymbol{\gamma}^{LP}$ achieves this bound and is therefore optimal:

$$(\boldsymbol{\gamma}^{LP})_t^k = \begin{cases} 1 & \text{if } t \in \mathcal{U}, k = K \\ 1 & \text{if } t \notin \mathcal{U}, k = 0 \\ 0 & \text{otherwise} \end{cases}$$

We then conclude that $\boldsymbol{p}^{LP} = \boldsymbol{p}^*$ is an optimal LP solution.

Proof of Step 2. Consider any feasible price vector $\boldsymbol{p}$ and let $\mathcal{A}$ be the set of promotions in $\boldsymbol{p}$. We next show that $POP(\boldsymbol{p}) \leq POP(\boldsymbol{p}^*)$ so that $\boldsymbol{p}^*$ is an optimal POP solution. If $\boldsymbol{p}$ uses the promotion $p_t = q^k$ during a period $t \notin \mathcal{U}$, then we can consider the reduced set of promotions $\mathcal{B} = \mathcal{A} \setminus \{(t,k)\}$. Note that the promotion (t,k) does not increase the profit at time t. Indeed, decreasing the price p_t will not increase the profit at time t since $f_t(p_t) = 1/q^0$ for all p_t, and potentially will reduce the profit in future periods $t+1, \ldots, t+M$. Thus, removing the promotion (t,k) increases the total profit, that is $POP(\boldsymbol{\gamma}(\mathcal{A})) \leq POP(\boldsymbol{\gamma}(\mathcal{B}))$ .By applying this procedure repeatedly, one can reach a price vector with only promotions in periods $t \in \mathcal{U}$ that achieves a profit at least equal to $POP(\boldsymbol{p})$. In other words, there exists an optimal POP solution with promotions only during periods $t \in \mathcal{U}$.

Proof of Step 3. Let $\boldsymbol{p}$ be a price vector that only contains promotions during periods $t \in \mathcal{U}$. Let n be the number of periods t in $\boldsymbol{p}$ such that $p_t = q^K$ ($n \leq L$ because $\mathcal{U}$ is composed of L periods). Note that all the successive promotions in $\mathcal{U}$ are separated by at least M periods so that each pair of promotions of $\boldsymbol{p}$ does not interact. Therefore, the profit of $\boldsymbol{p}$ is given by:

$$POP(\boldsymbol{p}) = POP(\boldsymbol{p}^0) + n(Z - Y) \leq POP(\boldsymbol{p}^0) + L(Z - Y).$$

From the definition of $\boldsymbol{p}^*$, we have that $POP(\boldsymbol{p}^*) = POP(\boldsymbol{p}^0) + L(Z - Y)$. Indeed, each promotion (t,K) of $\boldsymbol{p}^*$ results in an increase in profit of $Z - Y$, and each pair of promotions of

$\boldsymbol{p}^*$ is separated by at least M periods so that there is no interaction between promotions. Consequently, $\boldsymbol{p}^*$ is an optimal POP solution and the lower bound is tight. □

2. **Upper bound**

Proof. Let us denote the bound with n promotions by:

$$\underline{R}_n = \prod_{i=1}^{n-1} g_{i(S+1)}(q^K), \tag{EC.3}$$

when $\underline{R}_0 = 1$ by convention. We can also define the following limit:

$$\underline{R}_\infty = \lim_{n\to\infty} \underline{R}_n.$$

Note that $g_m(q^K) \le 1$ so that R_n is non-increasing with respect to n. Note also that $g_m(q^K) = 1$ for $m > M$ so that $\underline{R}_{M+1} = \underline{R}_{M+2} = \cdots = \underline{R}_\infty$, i.e., the sequence $\underline{R}_n$ converges.

In the case when $S \ge M$, we know from Proposition 1 that the LP approximation is exact. We also know from (EC.3) that $\underline{R}_n = 1$ for all n. Therefore, the result holds in this case.

We next consider that $S < M$ and define the following sequence of problems:

$$POP^n = POP(\{q^k\}_{k=0}^K, \{f_t^n\}_{t=1}^{T_n}, \{c_t\}_{t=1}^{T_n}, \{g_m\}_{m=1}^M, L_n, S),$$

where $\{q^k\}_{k=0}^K, \{g_m\}_{m=1}^M, S$ are given parameters and the costs $c_t = 0$. In addition, $L_n = n$, and $T_n = n(M+1)$. We choose the functions f_t^n to be equal:

$$f_t^n(p_t) = \begin{cases} Z/q^K & \text{if } 1 \le t \le LM+1 \text{ and } p_t = q^K, \\ 1/q^0 & \text{otherwise.} \end{cases}$$

where,

$$Y = 1 + \sum_{m=1}^{M} (1 - g_m(q^K)),$$
$$Z = 100Yn.$$

We prove the proposition by the following steps:

Step 1: We show that the following price vector is an optimal LP solution:

$$\boldsymbol{p}^{LP} = \Big(q^K, \underbrace{q^0, \ldots, q^0}_{S \text{ times}}, q^K, \underbrace{q^0, \ldots, q^0}_{S \text{ times}}, \ldots, q^K, \underbrace{q^0, \ldots, q^0}_{T-(L-1)(S+1)-1 \text{ times}}\Big).$$

Step 2: We show that:

$$POP^n(\boldsymbol{p}_n^{LP}) \le T - L + Z(\underline{R}_1 + \cdots + \underline{R}_n).$$

Step 3: We show the following lower bound for the optimal profit: $POP^n(\boldsymbol{p}_n^{POP}) \ge nZ$.

Step 4: We finally prove the convergence of the following limit, implying the desired result:

$$\lim_{n\to\infty} \frac{POP_n(\boldsymbol{p}_n^{POP})}{POP_n(\boldsymbol{p}_n^{LP})} = \frac{1}{\underline{R}_\infty}.$$

Proof of Step 1. Based on the above definitions, we have: $POP(\boldsymbol{p}\{(t,K)\}) = POP(\boldsymbol{p}^0) + Z - Y$ for $1 \le t \le LM+1$. Therefore, the LP coefficients are given by:

$$b_t^k = \begin{cases} Z - Y & \text{if } 1 \le t \le LM+1, k = K, \\ \le 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{U} = \{1, S+1, 2S+1, \ldots, LS+1\}$ denote the set of promotion periods in $\boldsymbol{p}^{LP}$.

Any LP optimal solution selects at most L of γ_t^k, for $k = 1, \ldots, K$ to be 1. Consequently, the optimal LP objective is bounded above by $T + L(Z - Y)$. In fact, the following $\boldsymbol{\gamma}^{LP}$ achieves this bound and is therefore optimal:

$$(\boldsymbol{\gamma}^{LP})_t^k = \begin{cases} 1 & \text{if } t \in \mathcal{U}, k = K \\ 1 & \text{if } t \notin \mathcal{U}, k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, we conclude that the price vector $\boldsymbol{p}^{LP}$ is an optimal LP solution.

Proof of Step 2. One can see that the profit induced by the i-th promotion of $\boldsymbol{p}_L^{LP}$ (at time $t = (i-1)S+1$) is $\underline{R}_i Z$ due to the effect of the promotions $1, 2, \ldots, (i-1)$. In addition, the profit from each non-promotion period is bounded above by 1. We obtain:

$$POP_n(\boldsymbol{\gamma}_n^{LP}) \le T - L + Z(\underline{R}_1 + \underline{R}_2 + \cdots + \underline{R}_n).$$

Proof of Step 3. Consider the following price vector:

$$\boldsymbol{p} = \Big(q^K, \underbrace{q^0, \ldots, q^0}_{M \text{ times}}, q^K, \underbrace{q^0, \ldots, q^0}_{M \text{ times}}, \ldots, q^K, \underbrace{q^0, \ldots, q^0}_{M \text{ times}}\Big).$$

Note that $\boldsymbol{p}$ is feasible for POP_n. Note that all the successive promotions are separated by at least M periods so that each pair of promotions of p does not interact. Therefore, the profit induced by the i-th promotion in $\boldsymbol{p}$ (at time $t = (i-1)M+1$) is Z. As a result, we obtain the following lower bound for the POP profit of $\boldsymbol{p}$:

$$POP_n(\boldsymbol{p}) \ge nZ.$$

This also provides us a lower bound for the optimal POP profit:

$$POP_n(\boldsymbol{p}_n^{POP}) \ge POP_n(\boldsymbol{p}) \ge nZ.$$

Proof of Step 4. We show that $\frac{1}{\underline{R}_\infty}$ is both a lower and upper bound of the limit. First, using Theorem 1 for POP^n, we have:

$$\frac{POP^n(\boldsymbol{p}_n^{POP})}{POP^n(\boldsymbol{p}_n^{LP})} \le \frac{1}{\underline{R}_n}.$$

By taking the limit when $n \to \infty$ on both sides:

$$\lim_{n\to\infty} \frac{POP^n(\boldsymbol{p}_n^{POP})}{POP^n(\boldsymbol{p}_n^{LP})} \leq \lim_{n\to\infty} \frac{1}{\underline{R}_n} = \frac{1}{\underline{R}_\infty}.$$

By using Steps 2 and 3, we obtain:

$$\begin{aligned}\lim_{n\to\infty} \frac{POP^n(\boldsymbol{p}_n^{POP})}{POP^n(\boldsymbol{p}_n^{LP})} &\geq \lim_{n\to\infty} \frac{n \cdot 100nY}{nM + 100nY(\underline{R}_1 + \underline{R}_2 + \cdots + \underline{R}_n)} \\ &= \lim_{n\to\infty} \frac{1}{\frac{M}{100nY} + \frac{\underline{R}_1 + \underline{R}_2 + \cdots + \underline{R}_n}{n}} = \frac{1}{\underline{R}_\infty}.\end{aligned}$$

In the last equality, we have used the fact that if $\{a_n\}_{n=1}^{\infty}$ converges to a finite limit a, then $\{\sum_{i=1}^{n} a_i/n\}_{n=1}^{\infty}$ also converges to a.

EC.4. Illustrating the bounds

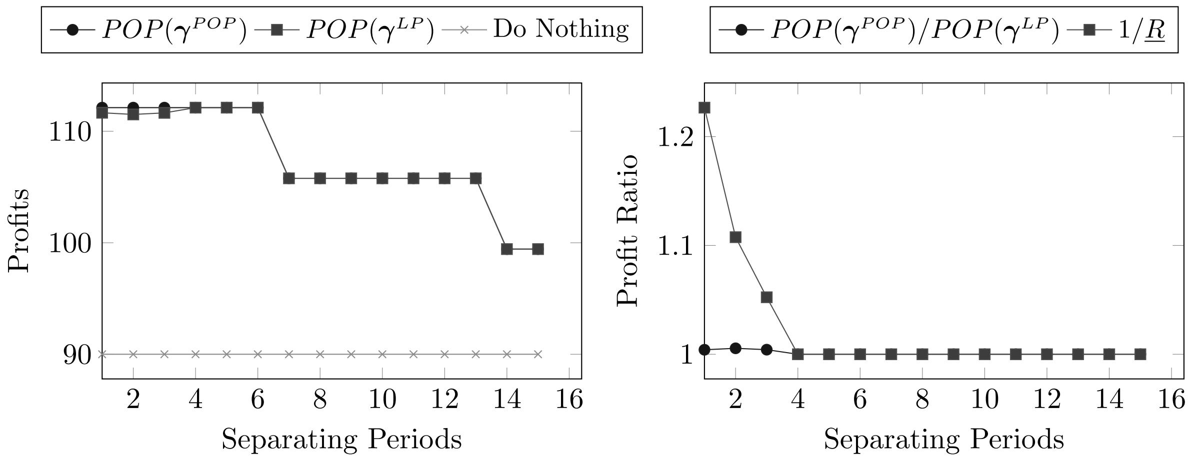
We show some examples that illustrate the behavior and quality of the bounds we have developed in the previous section. Recall that solving the POP can be hard in practice. Therefore, one can instead implement the LP solution. The resulting profit is then equal to $POP(\boldsymbol{\gamma}^{LP})$, whereas in theory, we could have obtained a maximum profit equal to the optimal POP profits denoted by $POP(\boldsymbol{\gamma}^{POP})$. In our numerical experiments, we examine the gap between $POP(\boldsymbol{\gamma}^{LP})$ and $POP(\boldsymbol{\gamma}^{POP})$ as a function of various parameters of the problem. In addition, we compare the ratio between $POP(\boldsymbol{\gamma}^{POP})$ and $POP(\boldsymbol{\gamma}^{LP})$ relative to the lower bound in Theorem 1 equal to $1/\underline{R}$. We also present an additional curve labeled “Do Nothing” as a benchmark (for which the no-promotion price is used at each time).

As we previously noted, the bounds we developed depend on four different parameters: the number of separating periods S, the number of promotions allowed L, the value of the minimum element of the price ladder q^K and the effect of past prices (i.e., the value of the memory parameter M as well as the magnitude of the functions g_k) . Below, we study the effect of each of these factors by varying them one at a time while the others are set to their worst case value.

All the figures below lead us to the following two observations: a) The LP solution achieves a profit that is close to the optimal profit. b) In particular, the actual optimality gap (between the POP objective at optimality versus evaluated at the LP approximation solution) seems to be of the order of 1-2 % and is smaller than the upper bound which we developed in Theorem 1.

In Figures EC.1, EC.2 and EC.3, the demand model we use is given by: $\log d_t(\boldsymbol{p}) = \log(10) - 4\log p_t + 0.5\log p_{t-1} + 0.3\log p_{t-2} + 0.2\log p_{t-3} + 0.1\log p_{t-4}$. In all the following tests, we decided to select $M = 4$. Using the data we have (for several stores and four categories of products), in our demand estimation ,the memory parameter M was always at most equal to 4. This value was not chosen arbitrarily and was estimated from data and tested out of sample. We tried to incorporate all the past prices (i.e., $p_{t-1}, p_{t-2}, \ldots, p_{t-T}$) in the regression model and observed that only the 4 first ones (or sometimes less than 4) were statistically significant, i.e., the p-value was less than 0.05 (similarly as in Table 2 in Section 7 of the paper). We then removed the non-significant observable variables and re-estimated the model parameters. In addition, we selected the minimal price q^K to be equal to 0.5, as it was the minimal value we observed in all our data sets.

Figure EC.1 Effect of varying the separating parameter S.



(a) Profits

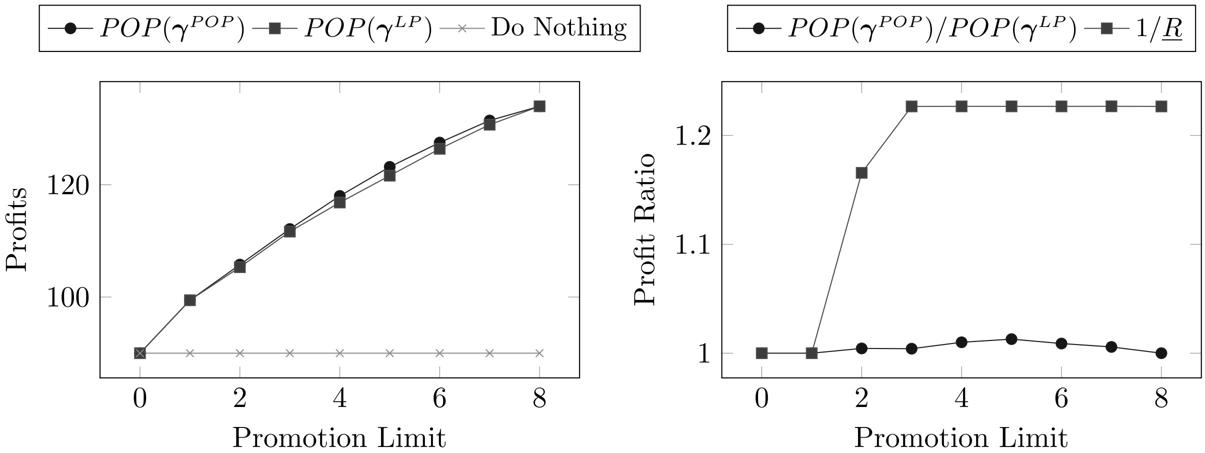
(b) Profit ratio

Note. Example parameters: $L = 3$, $\mathcal{Q} = \{1, 0.9, 0.8, 0.7, 0.6\}$.

Dependence on separating periods: In Figure EC.1, we vary the number of separating periods S from 1 to 16 (remember that the horizon is $T = 35$ weeks). We make the following observations: *a*) As one would expect from Proposition 1, the LP approximation coincides with the optimal POP solution when $S \geq M = 4$, i.e., $S \geq 4$. *b*) Our intuition suggests that as S increases, the upper bound $1/\underline{R}$ becomes better. Indeed, the promotions are further apart in time, reducing the interaction between promotions and improving the quality of the LP approximation. *c*) For values of $S \geq 1$, the upper bound is at most 23% in this example. In practice, typically the number of separating periods is at least 1 but often 2-4 weeks.

Dependence on the number of promotions allowed: In Figure EC.2, we vary the number of promotions allowed L between 0 and 8. We make the following observations: *a*) As one would expect from Proposition 1, the LP approximation coincides with the optimal POP solution when $L = 1$ (and of course $L = 0$). *b*) The upper bound is at most 23% in this example. Note that from the definition of $\underline{R}$ in equation (EC.3) of Theorem 1, $1/\underline{R}$ increases with L up to $L = 3$. Indeed, since $S = 1$ and $M = 4$, the first promotion can never interact with the fourth promotion or with further ones.

Figure EC.2 Effect of varying the number of promotions allowed L.

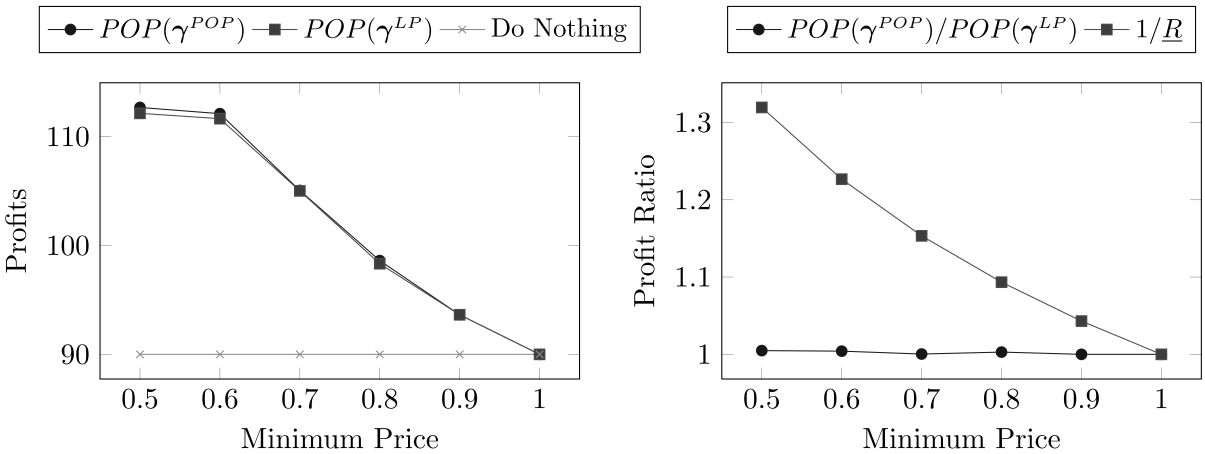


(a) Profits

(b) Profit ratio

Note. Example parameters: $S = 1, \mathcal{Q} = \{1, 0.9, 0.8, 0.7, 0.6\}$.

Figure EC.3 Effect of varying the minimum price q^K



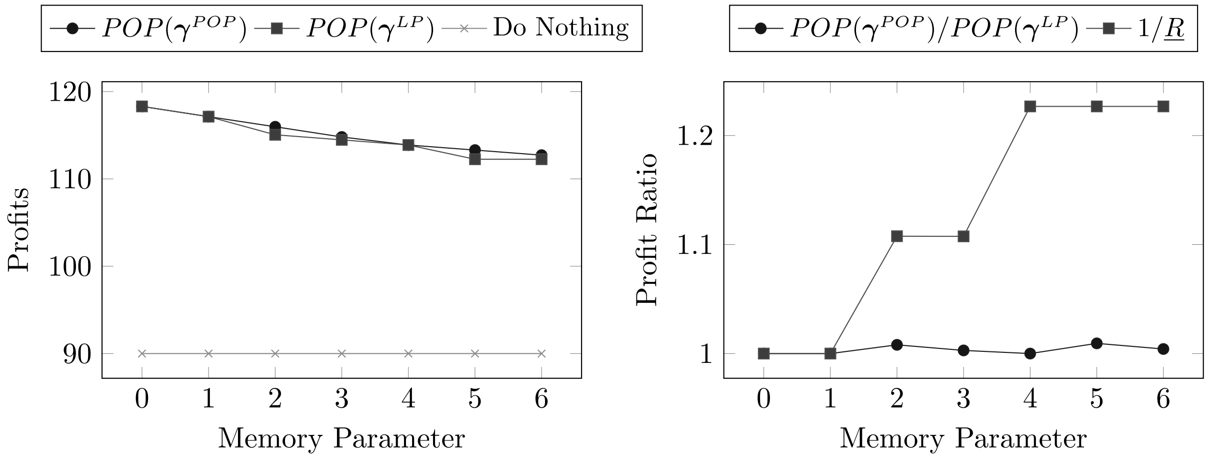
(a) Profits

(b) Profit ratio

Note. Example parameters: $L = 3, S = 1$.

Dependence on the minimal element of the price ladder: In Figure EC.3, we vary the (normalized) minimum promotion price q^K between 0.5 and 1. We make the following observations: *a*) As one would expect the LP approximation coincides with the optimal POP solution when $q^K = 1$, i.e., the promotion price is equal to the regular price so that promotions do not exist. *b*) The upper bound is 33% in this example for the case where a 50% promotion is allowed. If we restrict to a maximum of 30% promotion price, the bound becomes 14%. Using the definition of $\underline{R}$ from (EC.3), $1/\underline{R}$ decreases with q^K.

Figure EC.4 Effect of varying the memory parameter M



(a) Profits (b) Profit ratio

Note. Example parameters: $\log d_t(\boldsymbol{p}) = \log(10) - 4\log p_t + 0.2\log p_{t-1} + 0.2\log p_{t-2} + \cdots + 0.2\log p_{t-M}$; $L = 3, S = 1$.

Dependence on the length of the memory: In Figure EC.4, we vary the memory of costumers with respect to past prices, M between 0 and 6. Note that in this example, we have chosen the functions $g_1, g_2, \ldots, g_M$ to be equal. This choice can be seen as the "worst case" so that past prices have a uniformly strong effect on current demand. We make the following observations: *a*) As one would expect from Proposition 1, the LP approximation coincides with the optimal POP solution when $S \geq M$, i.e., $M \leq 1$. *b*) The upper bound is 23% in this example. Using the definition of $\underline{R}$ from (EC.3), $1/\underline{R}$ increases with M.

EC.5. Additive Demand

For some products, one may want to consider a demand model where the effect of past prices on current demand is additive. Therefore, we propose and study a class of additive demand functions. Suppose that past prices have an additive effect on current demand, so that the demand at time t is given by:

$$d_t = f_t(p_t) + g_1(p_{t-1}) + g_2(p_{t-2}) + \cdots + g_M(p_{t-M}). \tag{EC.4}$$

As we verify in Section 7 from the actual data, it is reasonable to assume the following structure for the functions g_k.

Assumption EC.1. 1. *The reduction effect is non-positive, i.e.,* $g_k(p) \leq 0$.

2. *Deeper promotions result in larger reduction in future demand, i.e.,* $p \leq q$ *implies that* $g_k(p) \leq g_k(q) \leq g_k(q^0) = 0$.
3. *The reduction effect is non-increasing with time since after the promotion:* g_k *is non-decreasing with respect to* k*, i.e.,* $g_k(p) \leq g_{k+1}(p)$.

Note that the above assumptions are analogous to Assumption 4 for the multiplicative model. We assume that for $k > M$, $g_k(p) = 0 \ \forall p$.

Remark. Equation (EC.4) represents a general class of demand functions, which admits as special cases several demand models used in practice. For example, the demand model used by Fibich et al. (2003) with symmetric reference price effects is given by:

$$d_t = a - \delta p_t - \phi(p_t - r_t). \tag{EC.5}$$

Equation (EC.5) can be rewritten as: $d_t = a - (\delta + \phi)p_t + \phi r_t$. Here, r_t represents the reference price at time t that consumers are forming based on their memory of past prices. The parameter ϕ denotes the price sensitivity with respect to the reference price, whereas $\delta + \phi$ represents the price sensitivity with respect to the current price. Note that the reference price at time t is given by:

$$r_t = (1-\theta)p_{t-1} + \theta r_{t-1},$$

and can be rewritten in terms of past prices as follows:

$$r_t = (1-\theta)p_{t-1} + \theta(1-\theta)p_{t-2} + \theta^2(1-\theta)p_{t-3} + \cdots = (1-\theta)\sum_{k=1}^{T}\theta^{k-1}p_{t-k},$$

where $0 \le \theta < 1$ denotes the memory of the consumers towards past prices. Therefore, the current demand from equation (EC.5) can be written as follows in terms of the current and past prices:

$$d_t = a - (\delta + \phi)p_t + \sum_{k=1}^{M=T}(1-\theta)\phi\ \theta^{k-1}p_{t-k}. \tag{EC.6}$$

One can see that equation (EC.6) falls under the model we proposed in (EC.4), when the functions g_k are chosen appropriately and the memory parameter M goes to infinity. In addition, the additive model from (EC.4) provides more flexibility in choosing the suitable memory parameter using data and allows us to give different weights depending on how far is the past promotion from the current time period.

Next, we present upper and lower bounds on the performance guarantee of the LP approximation relative to the optimal POP solution for the demand model in (EC.4).

EC.5.1. Bounds on Quality of Approximation

THEOREM EC.1. *Let $\boldsymbol{\gamma}^{POP}$ be an optimal solution to* (POP) *and let $\boldsymbol{\gamma}^{LP}$ be an optimal solution to the LP approximation. Then:*

$$1 \le \frac{POP(\boldsymbol{\gamma}^{POP})}{POP(\boldsymbol{\gamma}^{LP})} \le 1 + \frac{\overline{R}}{POP(\boldsymbol{\gamma}^{LP})}. \tag{EC.7}$$

where $\overline{R}$ is defined by:

$$\overline{R} = \sum_{i=1}^{\tilde{L}} \sum_{j=i+1}^{\tilde{L}} (q^K - q^0) g_{(j-i)(S+1)}(q^K). \tag{EC.8}$$

Proof. Note that the lower bound follows directly from the feasibility of $\boldsymbol{\gamma}^{LP}$ to the POP. We next prove the upper bound by showing the following chain of inequalities:

$$LP(\boldsymbol{\gamma}^{LP}) \overset{(i)}{\leq} POP(\boldsymbol{\gamma}^{LP}) \overset{(ii)}{\leq} POP(\boldsymbol{\gamma}^{POP}) \overset{(iii)}{\leq} LP(\boldsymbol{\gamma}^{POP}) + \overline{R} \overset{(iv)}{\leq} LP(\boldsymbol{\gamma}^{LP}) + \overline{R}. \tag{EC.9}$$

Inequalities (i) and (iii) follow from Proiposition EC.2 below. Inequality (ii) follows from the optimality of $\boldsymbol{\gamma}^{POP}$ and inequality (iv) follows from the optimality of $\boldsymbol{\gamma}^{LP}$. Therefore, we obtain:

$$1 = \frac{POP(\boldsymbol{\gamma}^{LP})}{POP(\boldsymbol{\gamma}^{LP})} \leq \frac{POP(\boldsymbol{\gamma}^{POP})}{POP(\boldsymbol{\gamma}^{LP})} \leq \frac{LP(\boldsymbol{\gamma}^{LP}) + \overline{R}}{POP(\boldsymbol{\gamma}^{LP})} \leq \frac{POP(\boldsymbol{\gamma}^{LP}) + \overline{R}}{POP(\boldsymbol{\gamma}^{LP})} = 1 + \frac{\overline{R}}{POP(\boldsymbol{\gamma}^{LP})}. \quad \square$$

The proof of Theorem EC.1 relies on the following result.

Proposition EC.2. *For a given promotion profile $\boldsymbol{\gamma}$, with the promotion set: $\{(t_1, k_1), \ldots, (t_n, k_n)\}$, the POP profits can be written as follows:*

$$POP(\boldsymbol{\gamma}_{\{(t_1,k_1),\ldots,(t_n,k_n)\}}) = LP(\boldsymbol{\gamma}_{\{(t_1,k_1),\ldots,(t_n,k_n)\}}) + ER(\boldsymbol{\gamma}_{\{(t_1,k_1),\ldots,(t_n,k_n)\}}). \tag{EC.10}$$

Here, $ER(\boldsymbol{\gamma}_{\{(t_1,k_1),\ldots,(t_n,k_n)\}})$ represents the error term between the POP and the LP objectives and is given by:

$$ER(\boldsymbol{\gamma}_{\{(t_1,k_1),\ldots,(t_n,k_n)\}}) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} (q^{k_j} - q^0) g_{t_j - t_i}(q^{k_i}). \tag{EC.11}$$

Consequently, for any feasible promotion profile $\boldsymbol{\gamma}$, the POP profits satisfies:

$$LP(\boldsymbol{\gamma}) \leq POP(\boldsymbol{\gamma}) \leq LP(\boldsymbol{\gamma}) + \overline{R}.$$

The proof of Proposition EC.2 can be found in Appendix EC.6. Proposition EC.2 states that the POP profits can be written as the sum of the LP approximation evaluated at the same promotion profile, plus some given error term that depends on the price differences and the functions $g_k(\cdot)$.

We next show that the POP profits are supermodular in promotions.

Corollary EC.1 **(Supermodularity of POP profits in promotions)**.
Let $A = \{(t_1, k_1), \ldots, (t_N, k_N)\}$ be a set of promotions with $1 \leq t_1 < t_2 < \cdots < t_n$ $(n \leq L)$ and let $B \subset A$. Consider a new promotion (t', k') where $t' \notin \{t_n\}_{n=1}^{N}$. Then, the new promotion (t', k') yields a greater marginal increase in profits when added to A than when added to B, that is:

$$POP(\boldsymbol{\gamma}_{A \cup \{(t',k')\}}) - POP(\boldsymbol{\gamma}_A) \geq POP(\boldsymbol{\gamma}_{B \cup \{(t',k')\}}) - POP(\boldsymbol{\gamma}_B). \tag{EC.12}$$

Proof. We first introduce the following definition. For two promotions (t,k) and (u,ℓ) with $t \neq u$, we define the interaction function:

$$\phi((t,k),(u,\ell)) = \begin{cases} (q^\ell - q^0) g_{u-t}(q^\ell) & \text{if } u > t; \\ (q^k - q^0) g_{t-u}(q^k) & \text{if } t > u. \end{cases}$$

Since $q^k, q^\ell \leq q^0$, and $g_m(p) \leq 0$ for all m and p, we have $\phi((t,k),(u,\ell)) \geq 0$. Observe that:

$$POP(\boldsymbol{\gamma}_{\{(t,k)\}}) = POP(\boldsymbol{\gamma}^0) + b_t^k,$$

where b_t^k are defined in (6) and represent the unilateral deviations in total profits by applying a single promotion at time t with price q^k. Similarly, we have: $POP(\boldsymbol{\gamma}_{\{(u,l)\}}) = POP(\boldsymbol{\gamma}^0) + b_u^\ell$. Therefore, we obtain:

$$POP(\boldsymbol{\gamma}_{\{(t,k),(u,\ell)\}}) = POP(\boldsymbol{\gamma}_{\{(t,k)\}}) + POP(\boldsymbol{\gamma}_{\{(u,\ell)\}}) - POP(\boldsymbol{\gamma}^0) + \phi((t,k),(u,\ell)).$$

In other words, the function $\phi((t,k),(u,\ell))$ compensates for the interaction term when we do both promotions (t,k) and (u,ℓ) simultaneously. From equation (EC.10) in Proposition EC.2, we obtain:

$$POP(\boldsymbol{\gamma}_A) = LP(\boldsymbol{\gamma}_A) + \sum_{(t,k),(u,\ell)\in A: t<u} (q^\ell - q^0) g_{u-t}(q^\ell)$$
$$POP(\boldsymbol{\gamma}_{A\cup\{(t',k')\}}) = LP(\boldsymbol{\gamma}_{A\cup\{(t',k')\}}) + \sum_{(t,k),(u,\ell)\in A\cup\{(t',k')\}: t<u} (q^\ell - q^0) g_{u-t}(q^\ell),$$

and similarly for the set B. By using the definition of the LP objective function:

$$LP(\boldsymbol{\gamma}_{\{(t_1,k_1),\dots,(t_n,k_n)\}}) = POP(\boldsymbol{\gamma}^0) + \sum_{i=1}^{n} \big(POP(\boldsymbol{\gamma}_{\{t_i,k_i\}}) - POP(\boldsymbol{\gamma}^0)\big),$$

we obtain: $LP(\boldsymbol{\gamma}_{A\cup\{(t',k')\}}) - LP(\boldsymbol{\gamma}_A) = POP(\boldsymbol{\gamma}') - POP(\boldsymbol{\gamma}^0)$ and: $LP(\boldsymbol{\gamma}_{B\cup\{(t',k')\}}) - LP(\boldsymbol{\gamma}_B) = POP(\boldsymbol{\gamma}') - POP(\boldsymbol{\gamma}^0)$, where we define $\boldsymbol{\gamma}' = \boldsymbol{\gamma}_{\{(t',k')\}}$. One can now obtain the following relations:

$$POP(\boldsymbol{\gamma}_{A\cup\{(t',k')\}}) - POP(\boldsymbol{\gamma}_A) = POP(\boldsymbol{\gamma}') - POP(\boldsymbol{\gamma}^0) + \sum_{(t,k)\in A} \phi((t,k),(t',k')),$$
$$POP(\boldsymbol{\gamma}_{B\cup\{(t',k')\}}) - POP(\boldsymbol{\gamma}_B) = POP(\boldsymbol{\gamma}') - POP(\boldsymbol{\gamma}^0) + \sum_{(t,k)\in B} \phi((t,k),(t',k')).$$

Therefore, we obtain:

$$\Big(POP(\boldsymbol{\gamma}_{A\cup\{(t',k')\}}) - POP(\boldsymbol{\gamma}_A)\Big) - \Big(POP(\boldsymbol{\gamma}_{B\cup\{(t',k')\}}) - POP(\boldsymbol{\gamma}_B)\Big)$$
$$= \sum_{(t,k)\in A\setminus B} \phi((t,k),(t',k')) \geq 0. \quad \square$$

Corollary EC.1 states that for an additive demand model as in (EC.4), the POP profits are supermodular in promotions. Note that unlike in the multiplicative case, the claim is valid for any set of promotions. Consequently, it supports intuitively the fact that the LP approximation underestimates the POP objective, i.e., $POP(\boldsymbol{\gamma}^{POP}) \geq LP(\boldsymbol{\gamma}^{POP})$. Note that by considering the objective (total profits) of problem (POP) as a continuous function of the prices $p_1, p_2, \ldots, p_T$, one can equivalently show the supermodularity property by checking the non-negativity of all the cross-derivatives. We next show that the upper and lower bounds of Theorem EC.1 are tight.

PROPOSITION EC.3 **(Tightness of the bounds for additive model)**.

1. *The lower bound in Theorem EC.1 is tight. More precisely, for any given price ladder, L, S and functions g_k, there exist T, costs c_t and functions f_t such that:*

$$POP(\boldsymbol{\gamma}^{POP}) = POP(\boldsymbol{\gamma}^{LP}).$$

2. *The upper bound in Theorem EC.1 is tight. More precisely, for any given price ladder, L, S and functions g_k, there exist T, costs c_t and functions f_t such that:*

$$POP(\boldsymbol{\gamma}^{POP}) = POP(\boldsymbol{\gamma}^{LP}) + \overline{R}.$$

The proof can be found in Appendix EC.7.

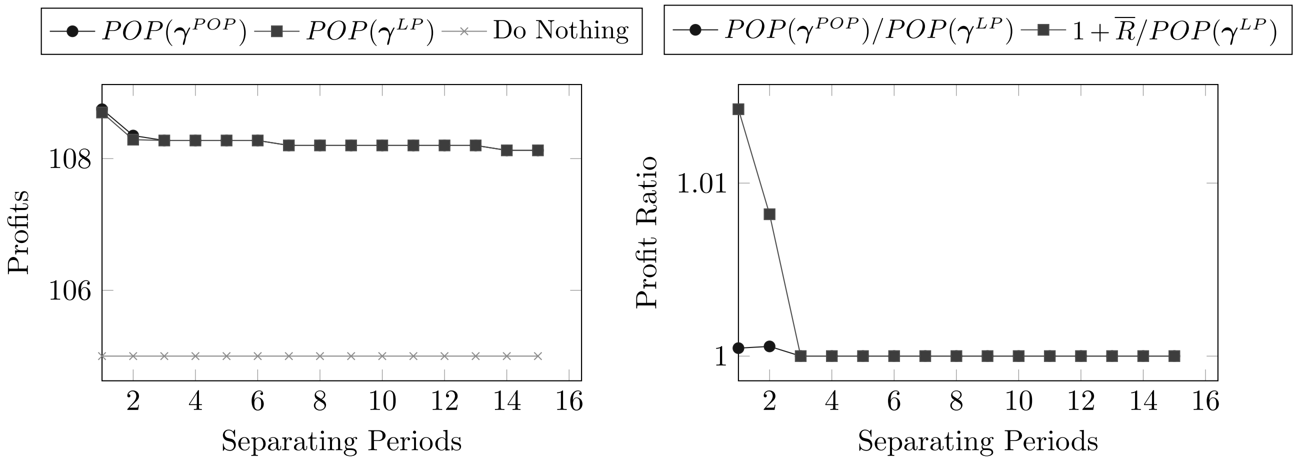
EC.5.2. Illustrating the bounds

In this section, we illustrate the bounds for the additive demand model by varying the different model parameters. We refer the reader to Section 5.2.2 for a discussion of the plots as a function of the various parameters since the trends we observe are similar in both the multiplicative and additive models.

In Figures EC.5, EC.6 and EC.7, the demand model is given by: $d_t(\mathbf{p}) = 30 - 50p_t + 15p_{t-1} + 10p_{t-2} + 5p_{t-3}$.

Dependence on separating periods: In Figure EC.5, we vary the number of separating periods S. We make the following observations: *a*) As one would expect from Proposition 1, the LP approximation coincides with the optimal POP solution for $S \geq M = 4$. *b*) As expected, as S increases, the upper bound $1 + \overline{R}/POP(\boldsymbol{\gamma}^{LP})$ decreases. Indeed, the larger is S, the more separated promotions are and as a result, it reduces the interaction between promotions which are neglected in the LP approximation. *c*) For any value of S, the upper bound on the relative optimality gap (between the POP objective at optimality versus evaluated at the LP approximation solution) is at most 2.5%, whereas the realized one is less than 1.5%. In practice, typically the number of separating periods is at least 2.

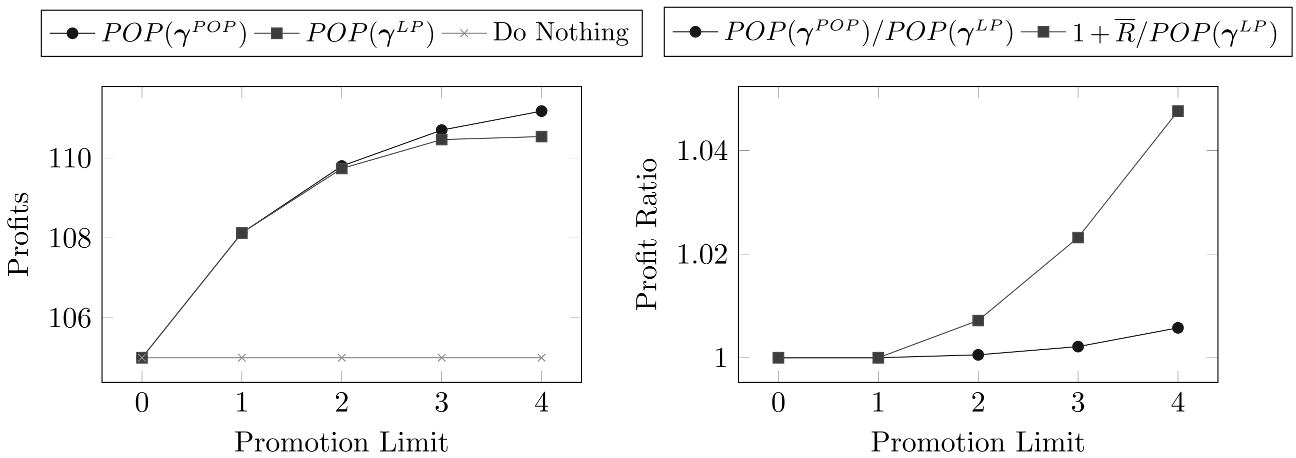
Figure EC.5 Effect of varying the separating periods S



(a) Profits (b) Profit ratio

Note. Example parameters: Example parameters: $L = 3, \mathcal{Q} = \{1, 0.95, 0.90, 0.85, 0.80, 0.75, 0.70\}$.

Figure EC.6 Effect of varying the promotion limit L

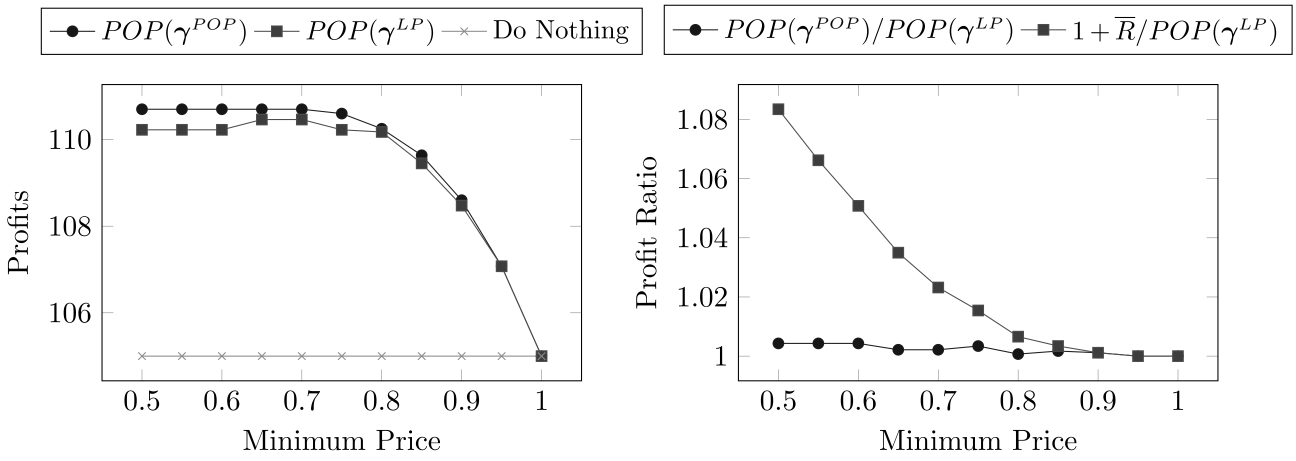


(a) Profits (b) Profit ratio

Note. Example parameters: $S = 0, \mathcal{Q} = \{1, 0.95, 0.90, 0.85, 0.80, 0.75, 0.70\}$.

Dependence on the number of promotions allowed: In Figure EC.6, we vary the number of promotions allowed L. We make the following observations: a) As one would expect from Proposition 1, the LP approximation coincides with the optimal POP solution for $L = 1$. b) For $L \leq 6$ (recall that $T = 13$), the upper bound on the relative optimality gap is at most 10%. As expected, the upper bound increases as L increases. This follows from the definition of $\overline{R}$ in Theorem EC.1. Unlike the multiplicative case for which $\underline{R}$ was asymptotically converging as L increases; in the additive case, $\overline{R}$ can grow to infinity as L increases.

Figure EC.7 Effect of varying the minimum price q^K

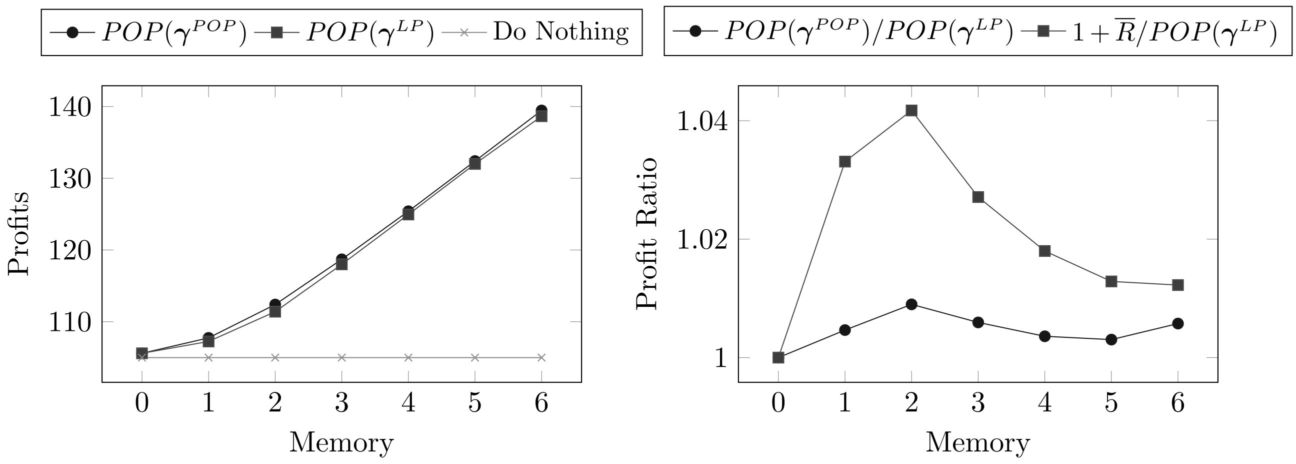


(a) Profits (b) Profit ratio

Note. Example parameters: $L=3, S=0$.

Dependence on the minimal price of the price ladder: In Figure EC.7, we vary the minimum promotion price q^K. We make the following observations: *a*) As one would expect, the LP approximation coincides with the optimal POP solution for $q^K=1$, i.e., the promotion price is equal to the regular price at all times. *b*) The upper bound on the relative optimality gap is at most 2.5%. From the definition of $\overline{R}$ in Theorem EC.1, one can see that the additive contribution $\overline{R}$ increases as q^K decreases.

Figure EC.8 Effect of varying the memory M



(a) Profits (b) Profit ratio

Note. Example parameters: $d_t(\mathbf{p}) = 30-(20+20M)p_t + 20p_{t-1} + 20p_{t-2} + \cdots + 20p_{t-M}$, $L=3, S=0$.

Dependence on the length of the memory: In Figure EC.8, we vary the memory parameter M. Note that in this example, we have chosen equal coefficients for $g_1, g_2, \ldots, g_M$, as a "worst case" so that past prices have a uniformly strong effect on current demand. We make the following observations: *a*) As one would expect from Proposition 1, the LP approximation coincides with the optimal POP solution for $S \geq M$, i.e., $M = 0$. *b*) The upper bound on the relative optimality gap is at most 4.5%. From the definition of $\overline{R}$ in TheoremEC.1, one can see that $\overline{R}$ increases with M, until it hits the constraint on the limited number of promotions (in this case is $L = 3$). In particular, we have two cases. When $M < L$, increasing the memory parameter by one unit will increase $\overline{R}$. Indeed, from the definition of $\overline{R}$, some of the terms $g_{(j-i)(S+1)}(q^K)$ will switch from zero to a negative value. When $M > L$, increasing the memory parameter by one one will not increase $\overline{R}$. In this case, the terms $g_{(j-i)(S+1)}(q^K)$ do not change.

EC.6. Proof of Proposition EC.2

Proof. Without loss of generality, we consider the case with the costs equal to zero, i.e., $c_t = 0;\ \forall t$. We next show that both sides of equation (EC.10) at each time period t are equal. Let us define the quantities $e_t^i = g_{t_i - t}(q^{k_i})$ for $t > t_i$ that capture the demand reduction at time t due to the earlier promotion q^{k_i} at time t_i. Let LP_t and POP_t denote the LP approximation and POP objectives at time t respectively. Consider a price vector of the form: $\mathbf{p}_{\{(t_1,k_1),\ldots,(t_n,k_n)\}}$. The LP approximation evaluated at this price vector is given by:

$$LP(\mathbf{p}_{\{(t_1,k_1),\ldots,(t_n,k_n)\}}) = POP(\mathbf{p^0}) + \sum_{i=1}^{n} \left[q^{k_i} POP(\mathbf{p}_{(t_i,k_i)}) - POP(\mathbf{p^0}) \right].$$

The POP objective using the single promotion (t_i, k_i) is given by:

$$POP(\mathbf{p}_{(t_i,k_i)}) = q^0 f_1(q^0) + \cdots + q^0 f_{t_i-1}(q^0) + q^{t_i} f_{t_i}(q^{k_i}) + q^0 \left[f_{t_i+1}(q^0) + e_{t_i+1}^i \right] + \cdots + q^0 \left[f_T(q^0) + e_T^i \right].$$

In addition, we have: $POP(\mathbf{p^0}) = \sum_{t=1}^{T} q^0 f_t(q^0)$. We next divide the analysis depending whether a promotion occurs at time t or not.

Case 1: Time t is not a promotion period, so that t is between two consecutive promotion periods $t_i < t < t_{i+1}$ (or t is after the last promotion). In this case, we have: $POP_t = q^0 \left[f_t(q^0) + e_t^1 + \cdots + e_t^i \right]$. The LP objective at time t is given by:

$$LP_t = q^0 f_t(q^0) + \sum_{j=1}^{i} \left(q^0 \left[f_t(q^0) + e_t^j \right] - q^0 f_t(q^0) \right) = q^0 \left[f_t(q^0) + e_t^1 + \cdots + e_t^i \right]. \tag{EC.13}$$

As a result, at each time t without a promotion, we have $POP_t = LP_t$ and hence equation (EC.10) is satisfied. We next consider the second case.

Case 2: Time t is a promotion period, i.e., $t = t_i$ for some i. In this case, we obtain:

$$POP_t = q^{k_i}\left[f_{t_i}(q^{k_i}) + e^1_{t_i} + \cdots + e^{i-1}_{t_i}\right]. \tag{EC.14}$$

The LP objective at time t is composed of three different parts. First, if $t_j < t$, then the contribution of $POP(\mathbf{p}_{(t_j,k_j)})$ at time t is equal to: $q^0\left[f_t(q^0) + e^j_{t_i}\right]$. Second, if $t_j = t = t_i$, then the contribution of $POP(\mathbf{p}_{(t_j,k_j)})$ at time t is equal to: $q^{k_i} f_{t_i}(q^{k_i})$. Third, if $t_j > t$, then the contribution of $POP(\mathbf{p}_{(t_j,k_j)})$ at time t is the same as the contribution of $POP(\mathbf{p}^0)$ at time t. Therefore, in a similar way as in equation (EC.13), the LP objective at time t can be written as:

$$LP_t = \sum_{j=1}^{i-1} q^0 e^j_{t_i} + q^{k_i} f_{t_i}(q^{k_i}). \tag{EC.15}$$

By comparing equations (EC.14) and (EC.15), one can see that equation (EC.10) is satisfied and this concludes the proof of the first claim.

The second claim is a consequence of the first one. The first inequality follows from the facts that $q^{k_j} - q^0 \leq 0$ and $g_{t_j - t_i}(q^{k_j}) \leq 0$. The second inequality follows from the facts that $0 \geq q^{k_j} - q^0 \geq q^K - q^0$, and $t_j - t_i \geq (j-i)(S+1)$ (from the constraints on separating periods between successive promotions). By using the properties of the functions g_k from Assumption EC.1, we obtain: $0 \geq g_{t_j - t_i}(q^{k_j}) \geq g_{(j-i)(S+1)}(q^K)$. □

EC.7. Proofs of Tightness for Additive Demand

1. **Lower bound**

Proof. In the case when $S \geq M$, we know from Proposition 1 that an optimal solution of the LP is also an optimal solution of the POP. Thus, the result holds in this case.

In the case when $S < M$, we will construct a POP problem:

$$POP(\{q^k\}_{k=0}^K, \{f_t\}_{t=1}^T, \{c_t\}_{t=1}^T, \{g_m\}_{m=1}^M, L, S),$$

and a price vector $\boldsymbol{p}^*$, which we will show is both an LP optimal solution and a POP optimal solution. Let $T = L(M+1)$. Let us define the price vector $\boldsymbol{p}^*$ by:

$$p^*_t = \begin{cases} q^K & t \in \mathcal{U}, \\ q^0 & t \notin \mathcal{U}. \end{cases}$$

Let $\mathcal{U} = \{1, (M+1)+1, 2(M+1)+1, \ldots, (L-1)(M+1)+1\}$ denote the promotion periods of $\boldsymbol{p}^*$.

Let us define $Y = \sum_{i=1}^M |g_i(q^K)|$ and $Z = (L+1)q^0 Y / q^K$ and the demand functions f_t to be:

$$f_t(p_t) = \begin{cases} Z & \text{if } t \in \mathcal{U} \text{ and } p_t = q^K, \\ Y & \text{otherwise.} \end{cases}$$

Note that for any feasible price vector $\boldsymbol{p}$, the demand at each time is nonnegative. Let us define the costs $c_t = 0, \forall t = 1, \ldots, T$. We prove the proposition by the following steps:

Step 1: We show that an optimal LP solution is the price vector $\boldsymbol{p}^{LP} = \boldsymbol{p}^*$.

Step 2: We show that an optimal POP solution is the price vector $\boldsymbol{p}^{POP} = \boldsymbol{p}^*$.

Proof of Step 1 By defintion, we have: $POP(\boldsymbol{p}\{(t,K)\}) - POP(\boldsymbol{p}^0) = q^K Z - q^0 Y - q^0 Y$ for $t \in \mathcal{U}$. The first term is the period t profit of $POP(\boldsymbol{p}\{(t,K)\})$, the second term is the period t profit of $POP(\boldsymbol{p}^0)$, and the third term is the reduction in profit of periods $t+1,\dots,t+M$ of $POP(\boldsymbol{p}\{(t,K)\})$ due to the promotion in period t. Therefore, the LP coefficients as defined in (6) are:

$$b_t^k = \begin{cases} q^K Z - 2q^0 Y \geq 0 & \text{if } t \in \mathcal{U}, k = K \\ \leq 0 & \text{otherwise} \end{cases}$$

The LP optimal solution selects at most L of γ_t^k, for $k = 1,\dots,K$ to be 1. Consequently, the optimal LP objective is bounded above by $Tq^0 Y + L(q^K Z - 2q^0 Y)$. In fact, the following $\boldsymbol{\gamma}^*$ corresponding to $\boldsymbol{p}^*$ achieves this bound and is therefore optimal:

$$(\boldsymbol{\gamma}^*)_t^k = \begin{cases} 1 & \text{if } t \in \mathcal{U}, k = K \\ 1 & \text{if } t \notin \mathcal{U}, k = 0 \\ 0 & \text{otherwise} \end{cases}$$

We conclude that $\boldsymbol{p}^{LP} = \boldsymbol{p}^*$.

Proof of Step 2 We show that for any feasible price vector $\boldsymbol{p}$, we have $POP(\boldsymbol{p}^*) \geq POP(\boldsymbol{p})$. Observe that the POP profit for $\boldsymbol{p}^*$ is given by:

$$POP(\boldsymbol{p}^*) = Lq^K Z + (T - L)q^0 Y - Lq^0 Y.$$

In particular, the first term corresponds to the profit from the promotion periods $\mathcal{U}$ and the second term is the profit from the non-promotion periods $\mathcal{T} \setminus \mathcal{U}$ before promotions. Finally, the third term represents the reduction in profit during the non-promotion periods due to the promotions in $\mathcal{U}$.

Let POP_t be the $POP(\boldsymbol{p})$ profit at period t. If we promote at time $t \in \mathcal{U}$ using the price q^K, then $POP_t = q^K Z$ and otherwise, $POP_t \leq q^0 Y$. For any $\boldsymbol{p} \neq \boldsymbol{p}^*$, $\boldsymbol{p}$ has at most $L - 1$ promotions at the time periods $t \in \mathcal{U}$. Therefore, we obtain: $POP(\boldsymbol{p}) \leq (L-1)q^K Z + (T - L + 1)q^0 Y$. The first term results from the promotions during the periods in $\mathcal{U}$, whereas the second term comes from the non-promotion periods. One can see that:

$$\begin{aligned} POP(\boldsymbol{p}^*) - POP(\boldsymbol{p}) &= Lq^K Z + (T - L)q^0 Y - Lq^0 Y - \left[(L-1)q^K Z + (T - L + 1)q^0 Y\right] \\ &= q^K Z - (L+1)q^0 Y \geq 0, \end{aligned}$$

from the definition of Z. Therefore, $POP(\boldsymbol{p}^*) \geq POP(\boldsymbol{p})$ as desired. □

2. **Upper bound**

Proof. In the case when $S \geq M$, we know from Proposition 1 that an optimal solution of the LP is also an optimal solution of the POP. We also know from equation (EC.8) that $\overline{R} = 0$. Thus, the result holds in this case.

In the case when $S < M$, we will construct a POP problem:

$$POP(\{q^k\}_{k=0}^K, \{f_t\}_{t=1}^T, \{c_t\}_{t=1}^T, \{g_m\}_{m=1}^M, L, S),$$

an optimal LP price vector $\boldsymbol{p}^{LP}$, and an optimal POP price vector $\boldsymbol{p}^{POP}$, such that $POP(\boldsymbol{p}^{POP}) = POP(\boldsymbol{p}^{LP}) + \overline{R}$. Let $T = (M+1)L$. Let us define $Y = \sum_{i=1}^M |g_i(q^K)|$, $Z = (L+1)q^0 Y/q^K$ and the demand functions f_t to be:

$$f_t(p_t) = \begin{cases} Z & \text{if } 1 \le t \le LM+1 \text{ and } p_t = q^K, \\ Y & \text{otherwise.} \end{cases}$$

Note that for any feasible price vector $\boldsymbol{p}$, the demand at each time is nonnegative. We prove the proposition by the following steps:

Step 1: We show that the following price vector is an optimal LP solution:

$$\boldsymbol{p}^{LP} = \Big(q^K, \underbrace{q^0, \dots, q^0}_{M \text{ times}}, q^K, \underbrace{q^0, \dots, q^0}_{M \text{ times}}, \dots, q^K, \underbrace{q^0, \dots, q^0}_{M \text{ times}}\Big).$$

Step 2: We show that the following price vector is an optimal POP solution:

$$\boldsymbol{p}^{POP} = \Big(q^K, \underbrace{q^0, \dots, q^0}_{S \text{ times}}, q^K, \underbrace{q^0, \dots, q^0}_{S \text{ times}}, \dots, q^K, \underbrace{q^0, \dots, q^0}_{T-(L-1)(S+1)-1 \text{ times}}\Big).$$

Step 3: We show that $POP(\boldsymbol{p}^{POP}) = POP(\boldsymbol{p}^{LP}) + \overline{R}$ which concludes the proof.

Proof of Step 1. By definition, we have:

$$POP(\boldsymbol{p}\{(t,K)\}) - POP(\boldsymbol{p}^0) = q^K Z - q^0 Y - q^0 Y$$

for $t \in \mathcal{U}$. The first term is the period t profit of $POP(\boldsymbol{p}\{(t,K)\})$, the second term is the period t profit of $POP(\boldsymbol{p}^0)$, and the third term is the reduction in profit of periods $t+1, \dots, t+M$ of $POP(\boldsymbol{p}\{(t,K)\})$ due to the promotion in period t. Therefore, the LP coefficients as defined in (6) are:

$$b_t^k = \begin{cases} q^K Z - 2q^0 Y \ge 0 & \text{if } t \in \mathcal{U}, k = K \\ \le 0 & \text{otherwise} \end{cases}$$

The LP optimal solution selects at most L of γ_t^k, for $k = 1, \dots, K$ to be 1. Consequently, the optimal LP objective is bounded above by $Tq^0 Y + L(q^K Z - 2q^0 Y)$. In fact, the following $\boldsymbol{\gamma}^{LP}$ corresponding to $\boldsymbol{p}^{LP}$ achieves this bound and is therefore optimal:

$$(\boldsymbol{\gamma}^{LP})_t^k = \begin{cases} 1 & \text{if } t \in \mathcal{U}, k = K \\ 1 & \text{if } t \notin \mathcal{U}, k = 0 \\ 0 & \text{otherwise} \end{cases}$$

We conclude that $\boldsymbol{p}^{LP}$ is an optimal solution to LP. Note that because any two promotions are separated by at least M periods, $ER(\boldsymbol{p}^{LP}) = 0$ and then from Proposition EC.2:

$$POP(\boldsymbol{p}^{LP}) = LP(\boldsymbol{p}^{LP}). \tag{EC.16}$$

Proof of Step 2. By using Proposition EC.2, we know that for any feasible price vector $\boldsymbol{p}$: $POP(\boldsymbol{p}) = LP(\boldsymbol{p}) + ER(\boldsymbol{p})$. One can see that $LP(\boldsymbol{p}) \leq LP(\boldsymbol{p}^{POP})$. Indeed, we note that the price vector $\boldsymbol{p}^{POP}$ is also optimal for the LP by using a similar argument as for $\boldsymbol{p}^{LP}$. In other words, in this case, both $\boldsymbol{p}^{LP}$ and $\boldsymbol{p}^{POP}$ are optimal LP solutions. By using the definition of $\overline{R}$ from (EC.8), one can see that $ER(\boldsymbol{p}) \leq \overline{R}$ for all feasible $\boldsymbol{p}$. In other words, $\overline{R}$ corresponds to the largest possible error term. In addition, we have in this case: $ER(\boldsymbol{p}^{POP}) = \overline{R}$ by construction. Since $LP(\boldsymbol{p}) \leq LP(\boldsymbol{p}^{POP})$ and $ER(\boldsymbol{p}) \leq ER(\boldsymbol{p}^{POP})$ for any $\boldsymbol{p}$, we obtain $POP(\boldsymbol{p}) \leq POP(\boldsymbol{p}^{POP})$ for any $\boldsymbol{p}$ so that $\boldsymbol{p}^{POP}$ is an optimal POP solution. In addition, we have shown that:

$$POP(\boldsymbol{p}^{POP}) = LP(\boldsymbol{p}^{POP}) + \overline{R}. \tag{EC.17}$$

Proof of Step 3. In the proof of Step 2 we have shown that $LP(\boldsymbol{p}^{LP}) = LP(\boldsymbol{p}^{POP})$. Combining this equation with (EC.16) and (EC.17) gives us the desired result. □