## Appendix A: Summary of Notation

**Table 1 Summary of Notation**

| | |
|---|---|
| $P_i$ : | Platform $i$ |
| $q_{ij}$ : | Perceived quality of Platform $i$ $(i=1,2,...,n)$ for customer segment $j$ $(j=1,2,...,m)$ |
| $\kappa_j$ : | Price sensitivity of segment $j$ |
| $q_{xj}$ : | Perceived quality of the new joint service for customer segment $j$ |
| $p_i$ : | Price of $P_i$ without the new joint service |
| $\tilde{p}_i$ : | Price of $P_i$ with the new joint service |
| $\tilde{p}_x$ : | Price of the new joint service |
| $\Lambda_j$ : | Total arrival rate of customer segment $j$ |
| $d_{ij}$ : | Arrival rate of customer segment $j$ to platform $P_i$ without the new joint service |
| $\tilde{d}_{ij}$ : | Arrival rate of customer segment $j$ to platform $P_i$ with the new joint service |
| $\tilde{d}_{xj}$ : | Arrival rate of customer segment $j$ to the new joint service |
| $a_{ik}$ : | Attractiveness of Platform $i$ for worker type $k$ $(k=1,2,...,l)$ |
| $\eta_k$ : | Wage sensitivity of worker type $k$ |
| $a_{xk}$ : | Attractiveness of the new joint service for worker type $k$ |
| $w_i$ : | Wage of $P_i$'s workers without the new joint service |
| $\tilde{w}_i$ : | Wage of $P_i$'s workers with the new joint service |
| $\Gamma_k$: | Total number of workers of type $k$ |
| $s_{ik}$ : | Number of workers of type $k$ working for $P_i$ without the new joint service |
| $\tilde{s}_{ik}$ : | Number of workers of type $k$ working for $P_i$ with the new joint service |
| $\gamma$ : | Fraction of profit generated by the new joint service allocated to $P_1$ |
| $\tilde{\lambda}_i$ : | Total number of workers needed by $P_i$ (with coopetition) |
| $\beta_i$ : | Fixed share of the price allocated to workers under a fixed-commission rate at $P_i$ |
| $\tilde{n}$ : | Number of customers per service for the new joint service |

## Appendix B: Proof of Statements

### Auxiliary Lemma

Before presenting the proofs of our results, we state and prove an auxiliary lemma which is extensively used throughout this appendix.

LEMMA 2. *Define*

$$\bar{d}_{ij} := \frac{\Lambda_j \exp(q_{ij} - \kappa_j p_i)}{1 + \sum_{i'=1}^n \exp(q_{i'j} - \kappa_j p_{i'})} \text{ for all } i,j,$$
$$\text{and } \bar{d}_i := \sum_{j=1}^m \bar{d}_{ij} = \sum_{j=1}^m \frac{\Lambda_j \exp(q_{ij} - \kappa_j p_i)}{1 + \sum_{i'=1}^n \exp(q_{i'j} - \kappa_j p_{i'})} \text{ for all } i.$$

*Then, we have for* $i = 1,2,...,n$ *and* $i' \neq i$, $\partial_{p_i} \bar{d}_{ij} = -\kappa_j (1 - \bar{d}_{ij}/\Lambda_j)\bar{d}_{ij}$, $\partial_{p_i} \bar{d}_i = \sum_{j=1}^m \partial_{p_i} \bar{d}_{ij} = -\sum_{j=1}^m \kappa_j (1 - \bar{d}_{ij}/\Lambda_j)\bar{d}_{ij}$, $\partial_{p_{i'}} \bar{d}_{ij} = \kappa_j \bar{d}_{ij}\bar{d}_{i'j}/\Lambda_j$, *and* $\partial_{p_{i'}} \bar{d}_i = \sum_{j=1}^m \partial_{p_{i'}} \bar{d}_{ij} = \sum_{j=1}^m \kappa_j \bar{d}_{ij}\bar{d}_{i'j}/\Lambda_j$.

*Proof.* Since $\bar{d}_{ij} = \frac{\Lambda_j \exp(q_{ij} - \kappa_{ij} p_i)}{1 + \sum_{i'=1}^n \exp(q_{i'j} - \kappa_{i'j} p_{i'})}$, we have

$$\begin{aligned}
\partial_{p_i} \bar{d}_{ij} =& \Lambda_j \frac{-\kappa_j \exp(q_{ij} - \kappa_j p_i)[1 + \sum_{i'=1}^n \exp(q_{i'j} - \kappa_j p_{i'})] + \kappa_j [\exp(q_{ij} - \kappa_j p_i)]^2}{[1 + \sum_{i'=1}^n \exp(q_{i'j} - \kappa_j p_{i'})]^2} \\
=& - \frac{\Lambda_j \kappa_j \exp(q_{ij} - \kappa_j p_i)}{1 + \sum_{i'=1}^n \exp(q_{i'j} - \kappa_j p_{i'})} + \frac{\kappa_j}{\Lambda_j} \left( \frac{\Lambda_j \exp(q_{ij} - p_i)}{1 + \sum_{i'=1}^n \exp(q_{i'j} - \kappa_j p_{i'})} \right)^2 \\
=& - \kappa_j \bar{d}_{ij} + \kappa_j/\Lambda_j (\bar{d}_{ij})^2 = -\kappa_j (1 - \bar{d}_{ij}/\Lambda_j)\bar{d}_{ij}.
\end{aligned}$$

Hence,

$$\partial_{p_i} \bar{d}_i = \sum_{j=1}^m \partial_{p_i} \bar{d}_{ij} = -\sum_{j=1}^m \kappa_j (1 - \bar{d}_{ij}/\Lambda_j)\bar{d}_{ij}.$$

Analogously,

$$\begin{aligned}\partial_{p_{i'}}\bar{d}_{ij} =&\frac{\kappa_j\Lambda_j\exp(q_{i'j}-\kappa_j p_{i'})\exp(q_{ij}-\kappa_j p_i)}{[1+\sum_{i''=1}^n\exp(q_{i''j}-\kappa_j p_{i''})]^2}\\ =&\frac{\kappa_j}{\Lambda_j}\cdot\frac{\Lambda_j\exp(q_{ij}-\kappa_j p_i)}{1+\sum_{i''=1}^n\exp(q_{i''j}-\kappa_j p_{i''})}\cdot\frac{\exp(q_{i'j}-\kappa_j p_{i'})}{1+\sum_{i''=1}^n\exp(q_{i''j}-\kappa_j p_{i''})}\\ =&\kappa_j\bar{d}_{ij}\bar{d}_{i'j}/\Lambda_j.\end{aligned}$$

Thus, for $i\neq i'$,

$$\partial_{p_{i'}}\bar{d}_i=\sum_{j=1}^m\partial_{p_{i'}}\bar{d}_{ij}=\sum_{j=1}^m\kappa_j\bar{d}_{ij}\bar{d}_{i'j}/\Lambda_j.\quad\square$$

**Proof of Lemma 1**

For each $i\in 1,2,...,n$, we define the following:

$$f_i(d_i,s_i)=\sum_{j=1}^m\frac{\Lambda_j\exp[\nu_j+\min\{1,s_i/d_i\}(q_{ij}-\kappa_j p_i-\nu_j)]}{1+\sum_{i'=1}^n\exp[\nu_j+\min\{1,s_{i'}/d_{i'}\}(q_{i'j}-\kappa_j p_{i'}-\nu_j)]},$$

and

$$g_i(d_i,s_i)=\sum_{k=1}^l\frac{\Gamma_k\exp[\omega_k+\min\{1,d_i/s_i\}(a_{ik}+\eta_k w_i-\omega_k)]}{1+\sum_{i'=1}^n\exp[\omega_k+\min\{1,d_{i'}/s_{i'}\}(a_{i'k}+\eta_k w_{i'}-\omega_k)]}.$$

It suffices to show that, for each $i$, there exists $(d_i,s_i)$ such that

$$\begin{cases}d_i=f_i(d_i,s_i)\\ s_i=g_i(d_i,s_i).\end{cases}$$

We next show that, given $s_i$, there exists a unique $d_i(s_i)$ increasing in $s_i$, such that $d_i(s_i)=f_i(d_i(s_i),s_i)$. One should note that $\exp[\nu_j+\min\{1,s_i/d_i\}(q_{ij}-\kappa_j p_i-\nu_j)]$ is continuously decreasing in $d_i$ for any $s_i$. Hence, $f_i(d_i,s_i)$ is also continuously decreasing in $d_i$. Furthermore, $f_i(0+,s_i)>0$ and $f_i(+\infty,s_i)=0$. Therefore, there exists a unique $d_i(s_i)$ such that $d_i(s_i)=f_i(d_i(s_i),s_i)$. To show that $d_i(s_i)$ is increasing in $s_i$, we observe that $f_i(d_i,s_i)$ is increasing in $s_i$ for any $d_i$. For $\hat{s}_i>s_i$, $d_i(s_i)=f_i(d_i(s_i),s_i)\le f_i(d_i(s_i),\hat{s}_i)$, which implies that $d_i(\hat{s}_i)\ge d_i(s_i)$, i.e., $d_i(s_i)$ increasing in $s_i$. The exact same argument implies that, given $d_i$, there exists a unique $s_i(d_i)$ increasing in $d_i$, such that $s_i(d_i)=g_i(d_i,s_i(d_i))$. Tarski's Fixed Point Theorem (see, e.g., Milgrom and Roberts 1990) suggests that there exists $(d_i,s_i)$ such that $d_i=f_i(d_i,s_i)$ and $s_i=g_i(d_i,s_i)$.

We now show that, for $\hat{s}_i>s_i$, $d_i(\hat{s}_i)-d_i(s_i)<\hat{s}_i-s_i$. Denote $\delta:=\hat{s}_i-s_i$. It is straightforward to check that $f_i(d_i(s_i)+\delta,s_i+\delta)<d(s_i)+\delta$. Thus, $d(\hat{s}_i)=d(s_i+\delta)<d_i(s_i)+\delta$, i.e., $d_i(\hat{s}_i)-d_i(s_i)<\hat{s}_i-s_i$. Analogously, we have for $\hat{d}_i>d_i$, $s_i(\hat{d}_i)-s_i(d_i)<\hat{d}_i-d_i$.

Finally, we show the uniqueness of $(d_i,s_i)$, such that $d_i=f_i(d_i,s_i)$ and $s_i=g_i(d_i,s_i)$. If there exist distinct $(d_i^1,s_i^1)$ and $(d_i^2,s_i^2)$ such that $d_i^j=f_i(d_i^j,s_i^j)$ and $s_i^j=g_i(d_i^j,s_i^j)$ for $j=1,2$, then we have $d_i^j=d_i(s_i^j)$ and $s_i^j=s_i(d_i^j)$ for $j=1,2$. Therefore,

$$|d_i^1-d_i^2|+|s_i^1-s_i^2|=|d_i(s_i^1)-d_i(s_i^2)|+|s_i(d_i^1)-s_i(d_i^2)|<|s_i^1-s_i^2|+|d_i^1-d_i^2|,$$

which leads to a contradiction. Thus, we must have $(d_i^1,s_i^1)=(d_i^2,s_i^2)$, so that there exists a unique $(d_i,s_i)$ such that $d_i=f_i(d_i,s_i)$ and $s_i=g_i(d_i,s_i)$. This completes the proof. $\square$

**Proof of Theorem 1**

We first introduce some notation that will prove useful in our analysis. Given the competitors' strategy $(p_{-i}, w_{-i})$, we define $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ as $P_i$'s best price and wage responses. We also define the best-response mapping of the two-sided competition game as

$$T(p,w) := \Big(p_i(p_{-i}, w_{-i}), w_i(p_{-i}, w_{-i}) : 1 \le i \le n\Big).$$

We then iteratively define the $k$-fold best-response mapping ($k \ge 2$) as

$$T^{(k)}(p,w) = \Big(p_i^{(k)}(p_{-i}, w_{-i}), w_i^{(k)}(p_{-i}, w_{-i}) : 1 \le i \le n\Big),$$

where for $i = 1, 2, ..., n$

$$p_i^{(k)}(p_{-i}, w_{-i}) = p_i\Big(p_1^{(k-1)}(p_{-1}, w_{-1}), w_1^{(k-1)}(p_{-1}, w_{-1}), ..., p_n^{(k-1)}(p_{-n}, w_{-n}), w_n^{(k-1)}(p_{-n}, w_{-n})\Big),$$
$$w_i^{(k)}(p_{-i}, w_{-i}) = p_i\Big(p_1^{(k-1)}(p_{-1}, w_{-1}), w_1^{(k-1)}(p_{-1}, w_{-1}), ..., p_n^{(k-1)}(p_{-n}, w_{-n}), w_n^{(k-1)}(p_{-n}, w_{-n})\Big).$$

We use $||\cdot||_1$ to represent the $\ell_1$ norm, that is, $||x||_1 = \sum_{i=1}^n |x_i|$ for $x \in \mathbb{R}^n$. The proof of Theorem 1 is based on the following four steps:

- Step I. Under equilibrium, $s_i^* = d_i^*$ for $i = 1, 2, ..., n$.
- Step II. The best-response functions $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are continuously increasing in $p_{-i}$ and $w_{-i}$. This will imply that an equilibrium exists.
- Step III. There exists a $k^*$, such that the $k^*$-fold best response is a contraction mapping under the $\ell_1$ norm, i.e., there exists a constant $\theta \in (0,1)$, such that

$$||T^{(k^*)}(p,w) - T^{(k^*)}(p', w')||_1 \le \theta ||(p,w) - (p', w')||_1.$$

This will imply that the equilibrium is unique.

- Step IV. For any $(p,w)$, the sequence $\{T^{(k)}(p,w) : k = 1, 2, ...\}$ converges to the unique equilibrium $(p^*, w^*)$ as $k \uparrow +\infty$. This will imply that the equilibrium can be computed using a *tatônnement* scheme.

Step I is proved by contradiction (see Lemma 3 below). We show that if $s_i^* > d_i^*$, then $P_i$ can unilaterally decrease $w_i$ to increase its profit; and if $s_i^* < d_i^*$, then $P_i$ can unilaterally increase $p_i$ to increase its profit. This implies that we must have $s_i^* = d_i^*$ under equilibrium.

Step II is proved by exploiting structural properties of the best-response functions $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$, and by using the fact that $d_i^* = s_i^*$ under equilibrium (see Lemma 4 below). Since the feasible region of $(p_{-i}, w_{-i})$ is a lattice, Step II immediately implies that an equilibrium exists by Tarski's Fixed Point Theorem.

Step III is proved by bounding the $\ell_1$ norm of $T(p,w)$. We note that $T(\cdot)$ is not necessarily a contraction mapping, but $T^{(k*)}(\cdot)$ for some $k^* > 1$ is (see Lemma 5 below). Using the result of Step III, a standard contradiction argument will show that the equilibrium is unique.

Step IV is proved by exploiting the contraction mapping property of $T^{(k)}(\cdot)$ (see Lemma 6 below). Putting Steps I–IV together concludes the proof of Theorem 1. □

The following lemma proves Step I in the proof of Theorem 1.

LEMMA 3. *Under equilibrium,* $d_i^* = s_i^*$ *for* $i = 1, 2$.

*Proof.* Assume by contradiction that $s_i^* < d_i^*$. This implies that $d_i^* > \min\{d_i^*, s_i^*\} = s_i^*$,

$$d_i^* = \sum_{j=1}^{m} \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i^*/d_i^*\}(q_{ij} - \kappa_j p_i^* - \nu_j)]}{1 + \sum_{i'=1}^{n} \exp[\nu_j + \min\{1, s_{i'}^*/d_{i'}^*\}(q_{i'j} - \kappa_j p_{i'}^* - \nu_j)]},$$

and

$$s_i^* = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_k + \min\{1, d_i^*/s_i^*\}(a_{ik} + \eta_k w_i^* - \omega_k)]}{1 + \sum_{i'=1}^{n} \exp[\omega_k + \min\{1, d_{i'}^*/s_{i'}^*\}(a_{i'k} + \eta_k w_{i'}^* - \omega_k)]}.$$

Consequently, $P_i$ can increase its price to $p_i^*(\epsilon) = p_i^* + \epsilon$ (for a sufficiently small $\epsilon > 0$) and $(w_i^*, p_{-i}^*, w_{-i}^*)$ remain unchanged, with the induced market outcome $(d_i^*(\epsilon), s_i^*(\epsilon), d_{-i}^*(\epsilon), s_{-i}^*(\epsilon))$, which satisfies

$$d_i^*(\epsilon) = \sum_{j=1}^{m} \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i^*(\epsilon)/d_i^*(\epsilon)\}(q_{ij} - \kappa_{ij}(p_i^* + \epsilon) - \nu_j)]}{1 + \exp[\nu_j + \min\{1, s_i^*(\epsilon)/d_i^*(\epsilon)\}(q_{ij} - \kappa_j(p_i^* + \epsilon) - \nu_j)] + \sum_{i' \neq i} \exp[\nu_j + \min\{1, s_{i'}^*(\epsilon)/d_{i'}^*(\epsilon)\}(q_{i'j} - \kappa_j p_{i'}^* - \nu_j)]}$$

and

$$s_i^*(\epsilon) = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_k + \min\{1, d_i^*(\epsilon)/s_i^*(\epsilon)\}(a_{ik} + \eta_k w_i^* - \omega_k)]}{1 + \sum_{i'=1}^{n} \exp[\omega_k + \min\{1, d_{i'}^*(\epsilon)/s_{i'}^*(\epsilon)\}(a_{i'k} + \eta_k w_{i'}^* - \omega_k)]}.$$

One can check that, for a sufficiently small $\epsilon > 0$, $s_i^*(\epsilon) < d_i^*(\epsilon) < d_i^*$, $s_i^*(\epsilon) \geq s_i^*$, and hence $\min\{d_i^*(\epsilon), s_i^*(\epsilon)\} = s_i^*(\epsilon)$, where the inequality follows from the fact that $d_i(\epsilon)$ and $s_i(\epsilon)$ are continuous in $\epsilon$. Thus, $\pi_i(\epsilon) = (p_i^* + \epsilon - w_i^*)\min\{d_i^*(\epsilon), s_i^*(\epsilon)\} > (p_i^* - w_i^*)s_i^* = \pi_i^*$, which contradicts the fact that $(p_i^*, w_i^*, p_{-i}^*, w_{-i}^*)$ is an equilibrium. Therefore, we must have $s_i^* \geq d_i^*$.

Assume by contradiction that $s_i^* > d_i^*$. This implies that $s_i^* > \min\{d_i^*, s_i^*\} = d_i^*$. Consequently, $P_i$ can decrease its wage to $w_i^*(\epsilon) = w_i^* - \epsilon$ (for a sufficiently small $\epsilon > 0$) and $(p_i^*, w_i^*, p_{-i}^*)$ remain unchanged, with the induced market outcome $(d_i^*(\epsilon), s_i^*(\epsilon), d_{-i}^*(\epsilon), s_{-i}^*(\epsilon))$, which satisfies

$$d_i^*(\epsilon) = \sum_{j=1}^{m} \frac{\Lambda_j \exp[\nu_j + \min\{1, s_i^*(\epsilon)/d_i^*(\epsilon)\}(q_{ij} - \kappa_j p_i^* - \nu_j)]}{1 + \exp[\nu_j + \min\{1, s_i^*(\epsilon)/d_i^*(\epsilon)\}(q_{ij} - \kappa_j p_i^* - \nu_j)] + \sum_{i' \neq i}^{n} \exp[\nu_j + \min\{1, s_{i'}^*(\epsilon)/d_{i'}^*(\epsilon)\}(q_{i'j} - \kappa_j p_{i'}^* - \nu_j)]},$$

and

$$s_i^*(\epsilon) = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_k + \min\{1, d_i^*(\epsilon)/s_i^*(\epsilon)\}(a_{ik} + \eta_k(w_i^* - \epsilon) - \omega_k)]}{1 + \exp[\omega_k + \min\{1, d_i^*(\epsilon)/s_i^*(\epsilon)\}(a_{i'k} + \eta_k(w_i^* - \epsilon) - \omega_k)] + \sum_{i' \neq i} \exp[\omega_k + \min\{1, d_{i'}^*(\epsilon)/s_{i'}^*(\epsilon)\}(a_{i'k} + \eta_k w_{i'}^* - \omega_k)]}.$$

One check that, for a sufficiently small $\epsilon > 0$, $s_i^* > s_i^*(\epsilon) > d_i^*(\epsilon) > d_i^*$, and hence $\min\{d_i^*(\epsilon), s_i^*(\epsilon)\} = d_i^*(\epsilon) > d_i^*$, where the inequality follows from the fact that $d_i(\epsilon)$ and $s_i(\epsilon)$ are continuous in $\epsilon$. Thus, $\pi_i(\epsilon) = (p_i^* - w_i^* + \epsilon)\min\{d_i^*(\epsilon), s_i^*(\epsilon)\} > (p_i^* - w_i^*)d_i^* = \pi_i^*$, contradicting that $(p_i^*, w_i^*, p_{-i}^*, w_{-i}^*)$ is an equilibrium. Therefore, we have $s_i^* \leq d_i^*$. Since $s_i^* \geq d_i^*$ and $s_i^* \leq d_i^*$, we conclude that $s_i^* = d_i^*$. □

The following lemma establishes Step II in the proof of Theorem 1.

LEMMA 4. $p_i(p_{-i}, w_{-i})$ *and* $w_i(p_{-i}, w_{-i})$ *are continuously increasing in* $p_{-i}$ *and* $w_{-i}$*. Hence, an equilibrium exists in the two-sided competition model.*

*Proof*. Since $s_i^*=d_i^*$, we denote $s=s_i=d_i$ as the demand/supply of $P_i$. Given $(p_{-i},w_{-i},s)$, we can formulate the price and wage optimization of $P_i$ as follows:

$$\begin{aligned}
&\max_{(p_i,w_i,s)} \pi_i(p_i,w_i,s|p_{-i},w_{-i})\\
&\text{where } \pi_i(p_i,w_i,s|p_{-i},w_{-i})=(p_i-w_i)s\\
&\qquad \sum_{j=1}^{m} d_{ij}=s\\
&\qquad p_i=\frac{q_{ij}}{\kappa_j}-\frac{1}{\kappa_j}\log\left(\frac{d_{ij}/\Lambda_j}{1-d_{ij}/\Lambda_j}\right)-\frac{1}{\kappa_j}\log\left(1+\sum_{i'\neq i}\exp[\nu_j+\min\{1,s_{i'}/d_{i'}\}(q_{i'j}-\kappa_j p_{i'}-\nu_j)]\right)\ \forall j\\
&\qquad \sum_{k=1}^{l} s_{ik}=s\\
&\qquad w_i=-\frac{a_{ik}}{\eta_k}+\frac{1}{\eta_{ik}}\log\left(\frac{s_{ik}/\Gamma_k}{1-s_{ik}/\Gamma_k}\right)+\frac{1}{\eta_k}\log\left(1+\sum_{i'\neq i}\exp[\omega_k+\min\{1,d_{i'}/s_{i'}\}(a_{i'k}+\eta_k w_{i'}-\omega_k)]\right)\ \forall k.
\end{aligned} \tag{4}$$

Since $p_i=\frac{q_{ij}}{\kappa_j}-\frac{1}{\kappa_j}\log\left(\frac{d_{ij}/\Lambda_j}{1-d_{ij}/\Lambda_j}\right)-\frac{1}{\kappa_j}\log\left(1+\sum_{i'\neq i}\exp[\nu_j+\min\{1,s_{i'}/d_{i'}\}(q_{i'j}-\kappa_j p_{i'}-\nu_j)]\right)$ for all $j$, then $d_{ij}$ is strictly decreasing in $p_i$ for all $j$. Together with $\sum_{j=1}^m d_{ij}=s$, it implies that given $s$, there exists a unique $p_i$ and a unique associated vector $(d_{i1},d_{i2},...,d_{im})$ that satisfy the constraints $p_i=\frac{q_{ij}}{\kappa_j}-\frac{1}{\kappa_j}\log\left(\frac{d_{ij}/\Lambda_j}{1-d_{ij}/\Lambda_j}\right)-\frac{1}{\kappa_j}\log\left(1+\sum_{i'\neq i}\exp[\nu_j+\min\{1,s_{i'}/d_{i'}\}(q_{i'j}-\kappa_j p_{i'}-\nu_j)]\right)$ for all $j$. Thus, given $(p_{-i},w_{-i})$ and $s$, there exists a unique price $p_i(s,p_{-i},w_{-i})$ that satisfies all the constraints in (4). Analogously, there exists a unique wage $w_i(s,p_{-i},w_{-i})$ that satisfies all the constraints in (4). The corresponding demand for $P_i$ from each segment $j$, $d_{ij}$, and the corresponding supply for $P_i$ from each worker type $k$, $s_{ik}$, are also uniquely determined. It is clear by (4) that given $s$, $p_i(s,p_{-i},w_{-i})$ is strictly increasing in $p_{i'}$ and that $w_i(s,p_{-i},w_{-i})$ is strictly increasing in $w_{i'}$ for all $i'\neq i$. In addition, given $(p_{-i},w_{-i})$, $p_i(s,p_{-i},w_{-i})$ is strictly decreasing in $s$, whereas $w_i(s,p_{-i},w_{-i})$ is strictly increasing in $s$. By calculating the cross derivative, we can show that $\pi_i(s|p_{-i},w_{-i}):=(p_i(s,p_{-i},w_{-i})-w_i(s,p_{-i},w_{-i}))s$ is supermodular in $(p_{i'},s)$ for any $i'\neq i$. Therefore, $s^*:=\arg\max_s \pi_i(s|p_{-i},w_{-i})$ is increasing in $p_{i'}$, which implies that $w_i(p_{-i},w_{-i})=w_i(s^*,p_{-i},w_{-i})$ is also increasing in $p_{i'}$ for any $i'\neq i$.

We next show that $p_i(p_{-i},w_{-i})=p_i(s^*,p_{-i},w_{-i})$ is also strictly increasing in $p_{i'}$ for $i'\neq i$. We define $m(s,p_{-i},w_{-i}):=p_i(s,p_{-i},w_{-i})-w_i(s,p_{-i},w_{-i})$ as $P_i$'s profit margin given $(s,p_{-i},w_{-i})$. Thus,

$$\pi_i'(s|p_{-i},w_{-i})=\partial_s m(s,p_{-i},w_{-i})s+m(s,p_{-i},w_{-i}).$$

Since $\pi_i'(s^*|p_{-i},w_{-i})=0$, we have $\partial_s m(p_{-i},w_{-i},s^*)s^*+m(p_{-i},w_{-i},s^*)=0$. One should note by (4) that $\partial_s m(p_{-i},w_{-i},s)s$ is strictly decreasing in $s$ and independent of $(p_{-i},w_{-i})$. Assume that $\hat{p}_{i'}>p_i$ $(i'\neq i)$, so we have $\hat{s}^*>s^*$. Thus, $\partial_s m(\hat{s}^*,\hat{p}_{-i},w_{-i})\hat{s}^*<\partial_s m(\hat{s}^*,p_{-i},w_{-i})s^*$. By the first-order condition (FOC), $\pi_i'(\hat{s}^*|\hat{p}_{-i},w_{-i})=\pi_i'(s^*|p_{-i},w_{-i})=0$, that is, $\partial_s m(\hat{s}^*,\hat{p}_{-i},w_{-i})\hat{s}^*+m(\hat{s}^*,\hat{p}_{-i},w_{-i})=\partial_s m(s^*,p_{-i},w_{-i})s^*+m(s^*,p_{-i},w_{-i})=0$. Hence, we must have $m(\hat{s}^*,\hat{p}_{-i},w_{-i})>m(s^*,p_{-i},w_{-i})$. Therefore

$$p_i(\hat{s}^*,\hat{p}_{-i},w_{-i})=w_i(\hat{s}^*,\hat{p}_{-i},w_{-i})+m_i(\hat{s}^*,\hat{p}_{-i},w_{-i})>w_i(s^*,p_{-i},w_{-i})+m_i(s^*,p_{-i},w_{-i})=p_i(s^*,p_{-i},w_{-i}).$$

Thus, both $p_i(p_{-i},w_{-i})$ and $w_i(p_{-i},w_{-i})$ are increasing in $p_{i'}$ $(i'\neq i)$. By using a similar argument, we can show that $s^*$ is decreasing in $w_{i'}$ $(i'\neq i)$, which further implies that $p_i(p_{-i},w_{-i})=p_i(s^*,p_{-i},w_{-i})$ is increasing in $w_{i'}$ for $i' \neq i$. Moreover, a similar first-order argument suggests that the profit margin $m(s^*, p_{-i}, w_{-i})$ is decreasing in $w_{i'}$ for $i' \neq i$. We then conclude that

$$w_i(p_{-i}, w_{-i}) = w_i(s^*, p_{-i}, w_{-i}) = p_i(s^*, p_{-i}, w_{-i}) - m_i(s^*, p_{-i}, w_{-i})$$

is increasing in $w_{i'}$ for $i' \neq i$. Thus, we have shown that both $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are increasing in $p_{i'}$ and in $w_{i'}$ for $i' \neq i$. The continuity of $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ follows from the fact that $\pi_i(s|p_{-i}, w_{-i})$ is continuous. This completes the proof of Step II. By Tarski's Fixed Point Theorem (see, e.g., Milgrom and Roberts 1990), the continuity and monotonicity of $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$, together with the fact that the feasible sets of $p_i(\cdot,\cdot)$ and $w_i(\cdot,\cdot)$ are lattices, imply that an equilibrium exists. □

We next show that the best response mapping is a contraction mapping, so that a unique equilibrium exists.

LEMMA 5. *There exists a $k^*$, such that the $k^*$-fold best response is a contraction mapping under the $\ell_1$ norm, that is, there exists a constant $\theta \in (0,1)$, such that*

$$||T^{(k^*)}(p, w) - T^{(k^*)}(p', w')||_1 \leq \theta ||(p, w) - (p', w')||_1.$$

*Furthermore, the equilibrium is unique.*

*Proof.* We assume that $(p, w)$ and $(\hat{p}, \hat{w})$ are identical except that $\hat{p}_{i'} = p_{i'} + \delta$ for some $i'$. We observe that, for any $i \neq i'$ and any $j$

$$\partial_{p_{i'}} \Big\{ -\frac{1}{\kappa_j} \log \Big[1 + \sum_{i'' \neq i} \exp[\nu_j + \min\{1, s_{i''}/d_{i''}\}(q_{i''j} - \kappa_j p_{i'} - \nu_j)]\Big]\Big\} \leq \frac{\exp(q_{i'j} - \kappa_j p_{i'})}{1 + \sum_{i'' \neq i} \exp(q_{i''j} - \kappa_j p_{i'})} < \frac{\exp(q_{i'j})}{1 + \exp(q_{ij})}.$$

By the mean-value theorem, for $\delta > 0$ and any $j$,

$$0 < \frac{1}{\kappa_j} \log \Big[1 + \sum_{i'' \neq i} \exp[\nu_j + \min\{1, \hat{s}_{i''}/\hat{d}_{i''}\}(q_{i''j} - \kappa_j \hat{p}_{i''} - \nu_j)]\Big] - \frac{1}{\kappa_j} \log \Big[1 + \sum_{i'' \neq i} \exp[\nu_j + \min\{1, s_{i''}/d_{i''}\}(q_{i''j} - \kappa_j p_{i''} - \nu_j)]\Big] < C_{i'j}\delta,$$

where $C_{i'j} := \frac{\exp(q_{i'j})}{1+\exp(q_{i'j})} < 1$. Similarly, we have, for $\delta > 0$ and $i \neq i'$ and any $k$,

$$0 < \frac{1}{\eta_k} \log \Big[1 + \sum_{i'' \neq i} \exp[\omega_k + \min\{1, \hat{d}_{i''}/\hat{s}_{i''}\}(a_{i''k} + \eta_k \hat{p}_{i''} - \omega_k)]\Big] - \frac{1}{\eta_k} \log \Big[1 + \sum_{i'' \neq i} \exp[\omega_k + \min\{1, d_{i''}/s_{i''}\}(a_{i''k} + \eta_k p_{i''} - \omega_k)]\Big] < D_{i'k}\delta,$$

where $D_{i'k} := \frac{\exp(a_{i'k})}{1+\exp(a_{i'k})} < 1$. Define $s_i^* := \arg\max_s \pi_i(s|p_{-i}, w_{-i})$ and $\hat{s}_i^* := \arg\max_s \pi_i(s|\hat{p}_{-i}, w_{-i})$ for $i \neq i'$. We denote the demand from each customer segment $j$ for $P_i$ associated with price vector $\hat{p}_{-i}$ (resp. $p_{-i}$) as $\hat{d}_{ij}^*$ (resp. $d_{ij}^*$). The supply of worker type $k$ for $P_i$ associated with price vector $\hat{p}_{-i}$ (resp. $p_{-i}$) is denoted as $\hat{s}_{ik}^*$ (resp. $s_{ik}^*$). Thus, we have $\sum_{j=1}^m \hat{d}_{ij}^* = \sum_{k=1}^l \hat{s}_{ik}^* = \hat{s}_i^*$ and $\sum_{j=1}^m d_{ij}^* = \sum_{k=1}^l s_{ik}^* = s_i^*$.

We denote $\delta_i^2 := \max_j \Big[\log\Big(\frac{\hat{d}_{ij}^*/\Lambda_j}{1-\hat{d}_{ij}^*/\Lambda_j}\Big) - \log\Big(\frac{d_{ij}^*/\Lambda_j}{1-s_{ij}^*/\Lambda_j}\Big)\Big] > 0$ and $\delta_i^3 := \max_k \Big[\log\Big(\frac{\hat{s}_{ik}^*/\Gamma_k}{1-\hat{s}_{ik}^*/\Gamma_k}\Big) - \log\Big(\frac{s_{ij}^*/\Gamma_k}{1-s_{ik}^*/\Gamma_k}\Big)\Big] > 0$. As shown in the proof of Step II of Theorem 1, $\hat{d}_{ij}^* > d_{ij}^*$ for all $j$ and $\hat{s}_{ik}^* > s_{ik}^*$ for all $k$, and $m_i(\hat{s}_i^*, \hat{p}_{-i}, w_{-i}) > m_i(s_i^*, p_{-i}, w_{-i})$, that is, for any $j$,

$$0 < [p_i(\hat{p}_{-i}, w_{-i}) - w_i(\hat{p}_{-i}, w_{-i})] - [p_i(p_{-i}, w_{-i}) - w_i(p_{-i}, w_{-i})] < \frac{1}{\kappa_j} \log \Big[1 + \sum_{i'' \neq i} \exp[\nu_j + \min\{1, \hat{s}_{i''}/\hat{d}_{i''}\}(q_{i''j} - \kappa_j \hat{p}_{i''} - \nu_j)]\Big]$$
$$- \frac{1}{\kappa_j} \log \Big[1 + \sum_{i'' \neq i} \exp[\nu_j + \min\{1, s_{i''}/d_{i''}\}(q_{i''j} - \kappa_j p_{i''} - \nu_j)]\Big] < C_{i'j}\delta. \quad (5)$$

Inequality (5) implies that $\delta_i^2 + \delta_i^3 < C_{i'j}\delta$ for any $j$. Therefore, we obtain

$$p_i(\hat{p}_{-i}, w_{-i}) - p_i(p_{-i}, w_{-i}) = -\log\Big(\frac{\hat{d}^*_{ij}/\Lambda_j}{1-\hat{d}^*_{ij}/\Lambda_j}\Big) + \log\Big(\frac{d^*_{ij}/\Lambda_j}{1-d^*_{ij}/\Lambda}\Big) +$$
$$\frac{1}{\kappa_j}\log\Big[1+\sum_{i''\neq i}\exp[\nu_j + \min\{1, \hat{s}_{i''}/\hat{d}_{i''}\}(q_{i''j} - \kappa_j \hat{p}_{i''} - \nu_j)]\Big] - \frac{1}{\kappa_j}\log\Big[1+\sum_{i''\neq i}\exp[\nu_j + \min\{1, s_{i''}/d_{i''}\}(q_{i''j} - \kappa_j p_{i''} - \nu_j)]\Big]$$
$$< C_{i'j}\delta - \delta_i^2.$$

Analogously, for all $k$, $w_i(\hat{p}_{-i}, w_{-i}) - w_i(p_{-i}, w_{-i}) = \log\Big(\frac{\hat{s}^*_{ik}/\Gamma_k}{1-\hat{s}^*_{ik}/\Gamma_k}\Big) - \log\Big(\frac{s^*_{ik}/\Gamma_k}{1-s^*_{ik}/\Gamma_k}\Big) = \delta_i^3 < D_{i'k}\delta - \delta_i^2$. As a result, for all $i \neq i'$ and any $j$ and $k$,

$$|p_i(\hat{p}_{-i}, w_{-i}) - p_i(p_{-i}, w_{-i})| < C_{i'j}\delta \text{ and } |w_i(\hat{p}_{-i}, w_{-i}) - w_i(p_{-i}, w_{-i})| < D_{i'k}\delta.$$

We define $p_i^{(k)}$ (resp. $w_i^{(k)}$) as the value of $p_i$ (resp. $w_i$) for the $k$-th iteration of $T$ operated on $(p, w)$. Analogously, $\hat{p}_i^{(k)}$ (resp. $\hat{w}_i^{(k)}$) as the value of $p_i$ (resp. $w_i$) for the $k$-th iteration of $T$ operated on $(\hat{p}, w)$. Repeating the argument above, we have that, for any $i$ and any $k \geq 1$,

$$|\hat{p}_i^{(k)} - p_i^{(k)}| < C^{k-1}C_{i'}\delta \text{ and } |\hat{w}_i^{(k)} - w_i^{(k)}| < D^{k-1}D_{i'}\delta,$$

where

$$C := \max\Big\{\frac{\exp(q_{ij})}{1+\exp(q_{ij})} : 1 \leq i \leq n, 1 \leq j \leq m\Big\} < 1 \text{ and } D := \max\Big\{\frac{\exp(a_{ik})}{1+\exp(a_{ik})} : 1 \leq i \leq n, 1 \leq k \leq l\Big\} < 1.$$

Define $(\hat{p}^{(k)}, \hat{w}^{(k)}) := (\hat{p}_i^{(k)}, \hat{w}_i^{(k)} : 1 \leq i \leq n)$ and $(p^{(k)}, w^{(k)}) := (p_i^{(k)}, w_i^{(k)} : 1 \leq i \leq n)$. We have, for any $k \geq 1$,

$$||(\hat{p}^{(k)}, \hat{w}^{(k)}) - (p^{(k)}, w^{(k)})||_1 \leq (C^{k-1}C_{i'} + D^{k-1}D_{i'})\delta < 2E^k\delta,$$

where $E := \max\{C, D\} < 1$. By using the triangular inequality, we have, for any $k \geq 1$,

$$||T^{(k)}(p, w) - T^{(k)}(p', w')||_1 \leq 2E^k ||(p, w) - (p', w')||_1.$$

We define $k^*$ as the smallest integer $k$ such that $2E^k < 1$ (i.e., the smallest integer $k$ such that $k > -\log(2)/\log(E)$). Therefore, we obtain

$$||T^{(k^*)}(p, w) - T^{(k^*)}(p', w')||_1 \leq 2E^{k^*}||(p, w) - (p', w')||_1 < \theta||(p, w) - (p', w')||_1,$$

where $\theta := 2E^{(k^*)} < 1$. We conclude that $T^{(k^*)}(\cdot, \cdot)$ is a contraction mapping under the $\ell_1$ norm.

We next show that the equilibrium is unique. Assume by contradiction that there are two distinct equilibria $(p^*, w^*)$ and $(\bar{p}^*, \bar{w}^*)$. Then, by the equilibrium definition, we have

$$T(p^*, w^*) = (p^*, w^*) \text{ and } T(\bar{p}^*, \bar{w}^*) = (\bar{p}^*, \bar{w}^*).$$

Therefore,

$$T^{(k^*)}(p^*, w^*) = (p^*, w^*) \text{ and } T^{(k^*)}(\bar{p}^*, \bar{w}^*) = (\bar{p}^*, \bar{w}^*).$$

Hence, we have

$$||T^{(k^*)}(p^*, w^*) - T^{(k^*)}(\bar{p}^*, \bar{w}^*)||_1 = ||(p^*, w^*) - (\bar{p}^*, \bar{w}^*)||_1. \tag{6}$$

Since $T^{(k^*)}(\cdot, \cdot)$ is a contraction mapping, we have

$$||T^{(k^*)}(p^*, w^*) - T^{(k^*)}(\bar{p}^*, \bar{w}^*)||_1 < \theta||(p^*, w^*) - (\bar{p}^*, \bar{w}^*)||_1,$$

contradicting Equation (6) if $(p^*, w^*) \neq (\bar{p}^*, \bar{w}^*)$. Thus, a unique equilibrium exists. □

The following lemma establishes Step IV in the proof of Theorem 1.

LEMMA 6. *$T^{(k)}(p,w)$ converges to the unique equilibrium as $k\uparrow+\infty$.*

*Proof.* It suffices to show that $T^{(k)}(p,w)$ converges to the equilibrium $(p^*,w^*)$ in the $(p,w)$ space as $k\uparrow+\infty$. As shown in Step III, $||T^{(k)}(p,w)-T^{(k)}(p',w')||_1\le 2E^k||(p,w)-(p',w')||_1$ for any $(p,w)$ and $(p',w')$. We define $(p^{(k)},w^{(k)}):=T^{(k)}(p,w)$ for $k\ge 1$. For any $k$ and $l>0$,

$$||(p^{(k)},w^{(k)})-(p^{(k+l)},w^{(k+l)})||_1\le\sum_{i=1}^{l}||(p^{(k+i)},w^{(k+i)})-(p^{(k+i-1)},w^{(k+i-1)})||_1$$

$$\le\sum_{i=1}^{l}2E^{(k+i-1)}||(p^{(1)},w^{(1)})-(p,w)||_1\le\sum_{i=1}^{+\infty}2E^{(k+i-1)}||(p^{(1)},w^{(1)})-(p,w)||_1=\frac{2||(p^{(1)},w^{(1)})-(p,w)||_1E^k}{1-E},$$

where the first inequality follows from the triangle inequality. Thus, $||(p^{(k)},w^{(k)})-(p^{(k+l)},w^{(k+l)})||_1\to 0$ uniformly with respect to $l$ as $k\uparrow+\infty$, that is, $\{(p^{(k)},w^{(k)}):k\ge 1\}$ is a Cauchy sequence, and hence $(p^{(k)},w^{(k)})$ converges to $(p^*,w^*)$, which is a fixed point of $T(\cdot,\cdot)$, namely, $T(p^*,w^*)=(p^*,w^*)$ so that $(p^*,w^*)$ is the unique equilibrium. Hence, the unique equilibrium can be obtained using a *tatônnement* scheme, and this concludes the proof of Theorem 1. □

**Proof of Proposition 1**

**Part (a)**. As shown in the proof of Theorem 1, the sequence $\{T^{(k)}(p^{m*},w^{m*}):k\ge 1\}$ converges to the equilibrium $(p^*,w^*)$. In the proof of Proposition 1, we have defined:

$$(p^{(k)},w^{(k)}):=T^{(k)}(p^{m*},w^{m*})\text{ for }k\ge 1,$$

and $(p^{(0)},w^{(0)}):=(p^{m*},w^{m*})$. We have also defined $s_i^{(k)}$ as the optimal demand/supply of $P_i$ in the $k$-th iteration of the *tatônnement* scheme. Then, it suffices to show that $p_i^{(k)}<p_i^{(m*)}$ and $w_i^{(k)}>w_i^{(m*)}$ for $k\ge 1$ and $i=1,2,...,n$.

Note that for a monopoly (i.e., a setting where a centralized decision maker seeks to maximize the total profit from all $n$ platforms), we have $d_i^{m*}=s_i^{m*}$ for $i=1,2$. Indeed, following the same argument as in the proof of Step I of Theorem 1, if $d_i^{m*}>s_i^{m*}$, we can increase $p_i$ and strictly increase the profit of each platform. Analogously, if $d_i^{m*}<s_i^{m*}$, we can increase $w_i$ and strictly increase the profit of each platform. As a result, under the optimal price and wage policies, $d_i^{m*}=s_i^{m*}$ for $i=1,2,...,n$.

We next show that $p_i^{(1)}<p_i^{(0)}$ and $w_i^{(1)}>w_i^{(0)}$ for all $i$. As shown in the proof of Theorem 1, $(p_i^{(1)},w_i^{(1)})$ can be represented by $\big(p_i(s_i^{(1)},p_{-i}^{(0)},w_{-i}^{(0)}),w_i(s_i^{(1)},p_{-i}^{(0)},w_{-i}^{(0)}\big)$, where $p_i(\cdot,\cdot,\cdot)$ (resp. $w_i(\cdot,\cdot,\cdot)$) is the price (resp. wage) policy of $P_i$ given $(s,p_{-i},w_{-i})$ and $s_i^{(1)}$ is the optimal supply (which is equal to demand) obtained by solving the following optimization problem:

$$\max_s\ \pi_i(s|p_{-i}^{(0)},w_{-i}^{(0)})$$

where $\pi_i(s|p_{-i},w_{-i})=(p_i-w_i)s$

$$\sum_{j=1}^{m}d_{ij}=s$$

$$p_i=\frac{q_{ij}}{\kappa_j}-\frac{1}{\kappa_j}\log\Big(\frac{d_{ij}/\Lambda_j}{1-d_{ij}/\Lambda_j}\Big)-\frac{1}{\kappa_j}\log\Big(1+\sum_{i'\neq i}\exp[\nu_j+\min\{1,s_{i'}/d_{i'}\}(q_{i'j}-\kappa_jp_{i'}-\nu_j)]\Big)\text{ for all }j$$

$$\sum_{k=1}^{l}s_{ik}=s$$

$$w_i=-\frac{a_{ik}}{\eta_k}-\frac{1}{\eta_{ik}}\log\Big(\frac{s_{ik}/\Gamma_k}{1-s_{ik}/\Gamma_k}\Big)+\frac{1}{\eta_k}\log\Big(1+\sum_{i'\neq i}\exp[\omega_k+\min\{1,d_{i'}/s_{i'}\}(a_{i'k}+\eta_kw_{i'}-\omega_k)]\Big)\text{ for all }k.$$

Under the optimal policy, we have $s_i^{m*} = d_i^{m*}$, so the optimal price and wage of a monopoly $(p_i^{m*}, w_i^{m*})$ can be obtained by $\big(p_i(s_i^{m*}, p_{-i}^{(0)}, w_{-i}^{(0)}), w_i(s_i^{m*}, p_{-i}^{(0)}, w_{-i}^{(0)})\big)$, where $s_i^{m*}$ is the solution to the following optimization problem:

$$\begin{aligned} &\max_s \Big[\pi_i(s|p_{-i}^{(0)}, w_{-i}^{(0)}) + \sum_{i'\neq i} \pi_{i'}(s)\Big] \\ \text{where } &\pi_{i'}(s) = (p_{i'}^{(0)} - w_{i'}^{(0)}) \min\{d_{i'}, s_{i'}\}, \ i' \neq i \\ \text{with } &d_{i'} = \sum_{j=1}^m \frac{\Lambda_j \exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]}{1 + \sum_{i''=1}^n \exp[\nu_j + \min\{1, s_{i''}/d_{i''}\}(q_{i''j} - \kappa_j p_{i''} - \nu_j)]} \\ &s_{i'} = \sum_{k=1}^l \frac{\Gamma_k \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k w_{i'} - \omega_k)]}{1 + \sum_{i''=1}^n \exp[\omega_k + \min\{1, d_{i''}/s_{i''}\}(a_{i''k} + \eta_k w_{i''} - \omega_k)]}. \end{aligned}$$

One can easily check that, for all $i' \neq i$, $d_{i'}$, $s_{i'}$, and $\pi_{i'}(\cdot)$ are all strictly decreasing in $s$. Since $s_i^{(1)}$ is the maximizer of $\pi_i(s)$, we must have $s_i^{m*} < s_i^{(1)}$. Since, by the Proof of Lemma 4, $p_i(s, p_{-i}^{(0)}, w_{-i}^{(0)})$ is strictly decreasing in $s$, whereas $w_i(s, p_{-i}^{(0)}, w_{-i}^{(0)})$ is strictly increasing in $s$, we have $p_i^{(1)} = p_i(s_i^{(1)}, p_{-i}^{(0)}, w_{-i}^{(0)}) < p_i(s_i^{(m*)}, p_{-i}^{(0)}, w_{-i}^{(0)})$ and $w_i^{(1)} = w_i(s_i^{(1)}, p_{-i}^{(0)}, w_{-i}^{(0)}) > w_i(s_i^{(m*)}, p_{-i}^{(0)}, w_{-i}^{(0)})$. Then, we have shown that $p_i^{(1)} < p_i^{(0)}$ and $w_i^{(1)} > w_i^{(0)}$ for all $i = 1, 2, ..., n$.

We next show that if $p_{i'}^{(k)} < p_{i'}^{(m*)}$ and $w_{i'}^{(k)} > w_{i'}^{(m*)}$ for any $i' \neq i$, then $p_i^{(k+1)} < p_i^{(m*)}$ and $w_i^{(k+1)} > w_i^{(m*)}$. Assume by contradiction that either $p_i^{(k+1)} \geq p_i^{(m*)}$ or $w_i^{(k+1)} \leq w_i^{(m*)}$. Then, we have $s_i^{(k+1)} < s_i^{m*}$ and $m_i^{(k+1)} := p_i^{(k+1)} - w_i^{(k+1)} > m_i^{m*} := p_i^{(m*)} - w_i^{(m*)}$. As shown in the proof of Theorem 1, $\partial_s m(s, p_{-i}, w_{-i})s$ is independent of $(p_{-i}, w_{-i})$ and decreasing in $s$. Thus, we have:

$$\partial_s \pi_i(s_i^{(k+1)}|p_{-i}^{(k)}, w_{-i}^{(k)}) = \partial_s m_i^{(k+1)} s_i^{(k+1)} + m_i^{(k+1)} > \partial_s m_i^{(m*)} s_i^{(m*)} + m_i^{(m*)} = \partial_s \pi_i(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}),$$

where the inequality follows from $s_i^{(k+1)} < s_i^{m*}$ and $m_i^{(k+1)} > m_i^{m*}$. By the FOC of the monopoly model,

$$\partial_s \pi_i(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) + \sum_{i'\neq i} \partial_s \pi_{i'}(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) = 0,$$

so, we have that

$$\partial_s \pi_i(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) = -\sum_{i'\neq i} \partial_s \pi_{i'}(s_i^{(m*)}|p_{-i}^{(m*)}, w_{-i}^{(m*)}) > 0,$$

where the inequality follows from the fact that $\pi_{i'}(\cdot|p_{-i}^{(m*)}, w_{-i}^{(m*)})$ is strictly decreasing in $s$ (for $i' \neq i$). This implies that $\partial_s \pi_i(s_i^{(k+1)}|p_{-i}^{(k)}, w_{-i}^{(k)}) > 0$, which contradicts the FOC $\partial_s \pi_i(s_i^{(k+1)}|p_{-i}^{(k)}, w_{-i}^{(k)}) = 0$. Thus, we must have $p_i^{(k+1)} < p_i^{(m*)}$ and $w_i^{(k+1)} < w_i^{(m*)}$ for all $i$. Proposition 1(a) then follows from taking the limit $p_i^* = \lim_{k\to+\infty} p_i^{(k)} < p_i^{(0)} = p_i^{m*}$ and $w_i^* = \lim_{k\to+\infty} w_i^{(k)} > w_i^{(0)} = w_i^{m*}$ for $i = 1, 2, ..., n$.

**Part (b)**. We first show that the best-response functions $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are increasing in $\Lambda_j$ for any $j = 1, 2, ..., m$. Recall from the proof of Theorem 1 that $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ can be characterized as the solution to the following optimization problem:

$$\max_s \ \pi_i(s|p_{-i}, w_{-i}, \Lambda_j)$$

where $\pi_i(s|p_{-i}, w_{-i}) = (p_i - w_i)s$

$$\sum_{j'=1}^{m} d_{ij'} = s$$

$$p_i = \frac{q_{ij'}}{\kappa_{j'}} - \frac{1}{\kappa_{j'}} \log\left(\frac{d_{ij'}/\Lambda_{j'}}{1-d_{ij'}/\Lambda_{j'}}\right) - \frac{1}{\kappa_{j'}} \log\left(1 + \sum_{i'\neq i} \exp[\nu_{j'} + \min\{1, s_{i'}/d_{i'}\}(q_{i'j'} - \kappa_{j'}p_{i'} - \nu_{j'})]\right) \text{ for all } j'$$

$$\sum_{k=1}^{l} s_{ik} = s$$

$$w_i = -\frac{a_{ik}}{\eta_k} - \frac{1}{\eta_k} \log\left(\frac{s_{ik}/\Gamma_k}{1-s_{ik}/\Gamma_k}\right) - \frac{1}{\eta_k} \log\left(1 + \sum_{i'\neq i} \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k w_{i'} - \omega_k)]\right) \text{ for all } k.$$

By computing the cross derivative, one can see that $\pi_i(s|p_{-i}, w_{-i}, \Lambda_j)$ is supermodular in $(s, \Lambda_j)$ for any $j$. Therefore, $s^*$ and

$$w_i(s, p_{-i}, w_{-i}) = \frac{a_{ik}}{\eta_k} - \frac{1}{\eta_k} \log\left(\frac{s_{ik}/\Gamma_k}{1-s_{ik}/\Gamma_k}\right) - \frac{1}{\eta_k} \log\left(1 + \sum_{i'\neq i} \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k w_{i'} - \omega_k)]\right) \text{ for all } k$$

are increasing in $\Lambda_j$ for any $i$ and $j$.

We define $t_{ij'} := \frac{d_{ij'}/\Lambda_{j'}}{1-d_{ij'}/\Lambda_{ij'}} = \frac{d_{ij'}}{\Lambda_{j'}-d_{ij'}}$. We then have $d_{ij'} = \frac{\Lambda_{j'}t_{ij'}}{1-t_{ij'}}$ and can write the following:

$$s^* = \max_s \ \pi_i(s|p_{-i}, w_{-i}, \Lambda_j)$$

where $\pi_i(s|p_{-i}, w_{-i}) = (p_i - w_i)s$

$$\sum_{j'=1}^{m} \frac{\Lambda_{j'}t_{ij'}}{1-t_{ij'}} = s$$

$$p_i = \frac{q_{ij'}}{\kappa_{j'}} - \frac{1}{\kappa_{j'}} \log\left(t_{ij'}\right) - \frac{1}{\kappa_{j'}} \log\left(1 + \sum_{i'\neq i} \exp[\nu_{j'} + \min\{1, s_{i'}/d_{i'}\}(q_{i'j'} - \kappa_{j'}p_{i'} - \nu_{j'})]\right) \text{ for all } j'$$

$$\sum_{k=1}^{l} s_{ik} = s$$

$$w_i = -\frac{a_{ik}}{\eta_k} - \frac{1}{\eta_{ik}} \log\left(\frac{s_{ik}/\Gamma_k}{1-s_{ik}/\Gamma_k}\right) + \frac{1}{\eta_k} \log\left(1 + \sum_{i'\neq i} \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k w_{i'} - \omega_k)]\right) \text{ for all } k.$$

If $\Lambda_j$ increases, it follows that $t^*_{ij'}$, which solves the above optimization problem decreases for all $j'$. Thus,

$$p_i(p_{-i}, w_{-i}) = \frac{q_{ij'}}{\kappa_{j'}} - \frac{1}{\kappa_{j'}} \log\left(t^*_{ij'}\right) - \frac{1}{\kappa_{j'}} \log\left(1 + \sum_{i'\neq i} \exp[\nu_{j'} + \min\{1, s_{i'}/d_{i'}\}(q_{i'j'} - \kappa_{j'}p_{i'} - \nu_{j'})]\right)$$

is increasing in $\Lambda_j$. We then have proved that both $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are increasing in $\Lambda_j$. Since $p_i(p_{-i}, w_{-i})$ and $w_i(p_{-i}, w_{-i})$ are both increasing in $p_{-i}$ and $w_{-i}$, then $p_i^{(k)}$ and $w_i^{(k)}$ are increasing in $\Lambda$ for any $k \geq 1$. By Theorem 1, $(p^*, w^*) = \lim_{k\uparrow+\infty}(p^{(k)}, w^{(k)})$. Thus, $p_i^* = \lim_{k\uparrow+\infty} p_i^{(k)}$ and $w_i^* = \lim_{k\uparrow+\infty} w_i^{(k)}$ for $i = 1, 2, ..., n$ are increasing in $\Lambda_j$ for any $j$. This concludes the proof of Proposition 1(b). □

**Proof of Theorem 2**

As in the proof of Theorem 1, we prove Theorem 2 using the following three steps:

- Under equilibrium, $s_i^{c*} \geq d_i^{c*}$, that is, supply dominates demand.

- The best-response price $p_i^c(p_{-i})$ is continuously increasing in $p_j$ for any $j \neq i$. By Tarski's Fixed Point Theorem, this monotonicity implies that an equilibrium exists.
- The best-response mapping $T^c(p) = (p_i^c(p_{-i}) : i = 1, 2, ..., n)$ satisfies

$$||T^c(p) - T^c(p')||_1 \leq q_c ||p - p'||_1 \text{ for some } q_c \in (0, 1).$$

This will imply that the equilibrium is unique and can be computed using a *tatônnement* scheme.

*Step I.* $s_i^{c*} \geq d_i^{c*}$

If $s_i^{c*} < d_i^{c*}$, then $P_i$ can increase its price from $p_i^{c*}$ to $\hat{p}_i^{c*} = p_i^{c*} + \epsilon$ (for a small $\epsilon > 0$), and accordingly its wage from $\beta_i p_i^{c*}$ to $\beta_i p_i^{c*} + \beta_i \epsilon$, where $\epsilon$ is small enough so that $\hat{s}_i^{c*} \leq \hat{d}_i$. With this price adjustment, $P_i$'s profit increases by at least $(1 - \beta_i)\epsilon s_i^{c*} > 0$, hence contradicting that $(p_i^{c*}, p_{-i}^{c*})$ is an equilibrium. Therefore, we must have $s_i^{c*} \geq d_i^{c*}$ for $i = 1, 2, ..., n$.

*Step II.* $p_i^c(p_{-i})$ *is continuously increasing in* $p_j$ *for all* $j \neq i$

Since $s_i^{c*} \geq d_i^{c*}$, the price/wage optimization of $P_i$ can be formulated as follows:

$$\begin{aligned}
&\max_{p_i} (1 - \beta_i) p_i d_i \\
&\text{s.t. } d_i = \sum_{j=1}^{m} \frac{\Lambda_j \exp(q_{ij} - \kappa_j p_i)}{1 + \exp(q_{ij} - \kappa_j p_i) + \sum_{i' \neq i} \exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]} \\
&\qquad s_i = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_j + (a_{ik} + \eta_j \beta_i p_i - \omega_j) \cdot d_i/s_i]}{1 + \exp[\omega_j + (a_{ik} + \eta_j \beta_i p_i - \omega_j) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_j + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_j \beta_{i'} p_{i'} - \omega_j)]} \\
&\qquad s_i \geq d_i.
\end{aligned}$$

Note that the objective function is supermodular in $(p_1, p_2, ..., p_n)$ and that the feasible set is a lattice. Thus, the best-response price $p_i^c(p_{-i})$ is continuously increasing in $p_{i'}$ for all $i' \neq i$. By Tarski's Fixed Point Theorem, an equilibrium $p^{c*}$ exists.

*Step III.* $T^c(\cdot)$ *is a contraction mapping under the* $\ell_1$ *norm*

As shown in the proof of Step II above,

$$\begin{aligned}
p_i^c(p_{-i}) = &\arg\max_{p_i} (1 - \beta_i) p_i d_i \\
&\text{s.t. } d_i = \sum_{j=1}^{m} \frac{\Lambda_j \exp(q_{ij} - \kappa_j p_i)}{1 + \exp(q_{ij} - \kappa_j p_i) + \sum_{i' \neq i} \exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]} \\
&\qquad s_i = \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_k + (a_{ik} + \eta_k \beta_i p_i - \omega_k) \cdot d_i/s_i]}{1 + \exp[\omega_k + (a_{ik} + \eta_k \beta_i p_i - \omega_k) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_k + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_k \beta_{i'} p_{i'} - \omega_k)]} \\
&\qquad s_i \geq d_i.
\end{aligned}$$

We define $\underline{p}_i(p_{-i})$ as the unconstrained optimizer of $p_i d_i$ (without the constraint $s_i \geq d_i$), which is increasing in $p_{i'}$ for each $i' \neq i$, as shown in Step II. We also define $\bar{p}_i(p_{-i})$ as the unique $p_i$ such that $s_i = d_i$, which is also increasing in $p_{i'}$ $(i' \neq i)$. We have $p_i^c(p_{-i}) = \max\{\underline{p}_i(p_{-i}), \bar{p}_i(p_{-i})\}$. It suffices to show that both $\underline{p}(\cdot) := (\underline{p}_1(\cdot), \underline{p}_2(\cdot), ..., \underline{p}_n(\cdot))$ and $\bar{p}(\cdot) := (\bar{p}_1(\cdot), \bar{p}_2(\cdot), ..., \bar{p}_n(\cdot))$ are contraction mappings under the $\ell_1$ norm. We next show that there exists a constant $C \in (0, 1)$, such that for any $p, p' \in \mathbb{R}_+^n$,

$$||\underline{p}(p) - \underline{p}(p')||_1 \leq C ||p - p'||_1 \text{ and } ||\bar{p}(p) - \bar{p}(p')||_1 \leq C ||p - p'||_1.$$

Since the MNL demand model satisfies the diagonal dominance condition, that is, for any $j$, $\frac{\partial^2 d_{ij}}{\partial p_i \partial p_{i'}} > 0$ for any $i' \neq i$, and

$$\frac{\partial^2 d_{ij}}{\partial (p_i)^2} < -\sum_{i' \neq i} \frac{\partial^2 d_{ij}}{\partial p_i \partial p_{i'}} < 0,$$

we have that, the $\ell_1$ matrix norm for the Jacobian of $\underline{p}(\cdot)$, denoted by $\underline{C}$, is strictly below 1 (i.e., $\underline{C} < 1$). Thus,

$$||\underline{p}(p) - \underline{p}(p')||_1 \leq \underline{C}||p - p'||_1. \tag{7}$$

We also note that $\bar{p}_i(p_{-i})$ satisfies the following equation:

$$\sum_{j=1}^{m} \frac{\Lambda_j \exp[q_{ij} - \kappa_j \bar{p}_i(p_{-i})]}{1 + \exp[q_{ij} - \kappa_j \bar{p}_i(p_{-i})] + \sum_{i' \neq i} \exp[\nu_j + \min\{1, s_{i'}/d_{i'}\}(q_{i'j} - \kappa_j p_{i'} - \nu_j)]}$$
$$= \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(p_{-i}) - \omega_j) \cdot d_i/s_i]}{1 + \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(p_{-i}) - \omega_j) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_j + \min\{1, d_{i'}/s_{i'}\}(a_{i'k} + \eta_j \beta_{i'} p_{i'} - \omega_j)]} := s.$$

If $\hat{p}_{i'} = p_{i'} + \delta$ for some $i' \neq i$ and $\delta > 0$ and $\hat{p}_{i''} = p_{i''}$ for all other $i'' \neq i$ and $i'' \neq i'$, we have

$$\sum_{j=1}^{m} \frac{\Lambda_j \exp[q_{ij} - \kappa_j \bar{p}_i(p_{-i})]}{1 + \exp[q_{ij} - \kappa_j \bar{p}_i(p_{-i})] + \sum_{i' \neq i} \exp[\nu_j + \min\{1, \hat{s}_{i'}/\hat{d}_{i'}\}(q_{i'j} - \kappa_j \hat{p}_{i'} - \nu_j)]} > s, \text{ whereas}$$

$$\sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(p_{-i}) - \omega_j) \cdot d_i/s_i]}{1 + \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(p_{-i}) - \omega_j) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_j + \min\{1, \hat{d}_{i'}/\hat{s}_{i'}\}(a_{i'k} + \eta_j \beta_{i'} \hat{p}_{i'} - \omega_j)]} < s.$$

We denote the induced supply and demand under the price vector $\hat{p}$ as $\hat{s}$. Furthermore, we have

$$\sum_{j=1}^{m} \frac{\Lambda_j \exp[q_{ij} - \kappa_j \bar{p}_i(\hat{p}_{-i})]}{1 + \exp[q_{ij} - \kappa_j \bar{p}_i(\hat{p}_{-i})] + \sum_{i' \neq i} \exp[\nu_j + \min\{1, \hat{s}_{i'}/\hat{d}_{i'}\}(q_{i'j} - \kappa_j \hat{p}_{i'} - \nu_j)]}$$
$$< \sum_{j=1}^{m} \frac{\Lambda_j \exp[q_{ij} - \kappa_j \bar{p}_i(p_{-i})]}{1 + \exp[q_{ij} - \kappa_j \bar{p}_i(p_{-i})] + \sum_{i' \neq i} \exp[\nu_j + \min\{1, \hat{s}_{i'}/\hat{d}_{i'}\}(q_{i'j} - \kappa_j \hat{p}_{i'} - \nu_j)]} := \bar{s}, \text{ and}$$

$$\sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(\hat{p}_{-i}) - \omega_j) \cdot d_i/s_i]}{1 + \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(\hat{p}_{-i}) - \omega_j) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_j + \min\{1, \hat{d}_{i'}/\hat{s}_{i'}\}(a_{i'k} + \eta_j \beta_{i'} \hat{p}_{i'} - \omega_j)]}$$
$$> \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(p_{-i}) - \omega_j) \cdot d_i/s_i]}{1 + \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(p_{-i}) - \omega_j) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_j + \min\{1, \hat{d}_{i'}/\hat{s}_{i'}\}(a_{i'k} + \eta_j \beta_{i'} \hat{p}_{i'} - \omega_j)]} =: \underline{s}.$$

Therefore,

$$\hat{s} = \sum_{j=1}^{m} \frac{\Lambda_j \exp[q_{ij} - \kappa_j \bar{p}_i(\hat{p}_{-i})]}{1 + \exp[q_{ij} - \kappa_j \bar{p}_i(\hat{p}_{-i})] + \sum_{i' \neq i} \exp[\nu_j + \min\{1, \hat{s}_{i'}/\hat{d}_{i'}\}(q_{i'j} - \kappa_j \hat{p}_{i'} - \nu_j)]}$$
$$> \sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(\hat{p}_{-i}) - \omega_j) \cdot d_i/s_i]}{1 + \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(\hat{p}_{-i}) - \omega_j) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_j + \min\{1, \hat{d}_{i'}/\hat{s}_{i'}\}(a_{i'k} + \eta_j \beta_{i'} \hat{p}_{i'} - \omega_j)]} \in (\underline{s}, \bar{s}).$$

If $\hat{s} < s$, define $p'$ as the solution to $\sum_{k=1}^{l} \frac{\Gamma_k \exp[\omega_j + (a_{ik} + \eta_j \beta_i p' - \omega_j) \cdot d_i/s_i]}{1 + \exp[\omega_j + (a_{ik} + \eta_j \beta_i \bar{p}_i(p_{-i}) - \omega_j) \cdot d_i/s_i] + \sum_{i' \neq i} \exp[\omega_j + \min\{1, \hat{d}_{i'}/\hat{s}_{i'}\}(a_{i'k} + \eta_j \beta_{i'} \hat{p}_{i'} - \omega_j)]} = s > \hat{s}$. Hence, $\bar{p}_i(\hat{p}_{-i}) < p'$. By the diagonal dominance property of the MNL demand model, we have $0 < \bar{p}(\hat{p}_{-i}) - \bar{p}(p_{-i}) < p' - \bar{p}(p_{-i}) < q_s \delta$, where $q_s := \max\{\frac{\exp(a_{ik})\beta_i}{1 + \exp(a_{ik})} : 1 \leq i \leq n, 1 \leq k \leq l\} < 1$.

Analogously, if $\hat{s} > s$, assume that $p''$ satisfies $\sum_{j=1}^{m} \frac{\Lambda_j \exp(q_{ij} - \kappa_j p'')}{1 + \exp(q_{ij} - \kappa_j p'') + \sum_{i' \neq i} \exp[\nu_j + \min\{1, \hat{s}_{i'}/\hat{d}_{i'}\}(q_{i'j} - \kappa_j \hat{p}_{i'} - \nu_j)]} = s < \hat{s}$. Since $\hat{s} > s$, we have $\bar{p}(\hat{p}_{-i}) < p''$. By the diagonal dominance condition of the MNL model, we have $0 < \bar{p}(\hat{p}_{-i}) - \bar{p}(p_{-i}) < p'' - \bar{p}(p_{-i}) < q_d \delta$, where $q_d := \max\{\frac{\exp(q_{ij})\beta_i}{1 + \exp(a_{ij})} : 1 \leq i \leq n, 1 \leq j \leq m\} < 1$.

We define $q_c := \max\{q_d, q_s\} < 1$. The above analysis implies that

$$||\bar{p}(p) - \bar{p}(p')||_1 \leq q_c||p - p'||_1 \tag{8}$$

By combining Equations (7) and (8), we obtain $||\bar{p}(p) - \bar{p}(p')||_1 \leq C||p - p'||_1$, where $C := \max\{\underline{C}, q_c\} < 1$. We have established that under a fixed commission, the best-response is a contraction mapping over the strategy space. Then, by using Banach's Fixed Point Theorem, a unique equilibrium exists and can be computed using a *tatônnement* scheme. This concludes the proof of Theorem 2. □

### B.1. Proof of Proposition 2

In the the proof of Proposition 2, we define:

$$(p^{(k)}, w^{(k)}) := T^{(k)}(p^{c*}, \beta p^{c*}) \text{ for } k \geq 1,$$

and $(p^{(0)}, w^{(0)}) = (p^{c*}, \beta p^{c*})$. By Theorem 1, we have $(p^{(k)}, w^{(k)})$ converges to $(p^*, w^*)$, as $k \uparrow +\infty$. Furthermore, by the symmetry of the model primitives, we have, if $p^{c*}$ is symmetric, $p_1^{(k)} = p_2^{(k)} = ... = p_n^{(k)}$ and $w_1^{(k)} = w_2^{(k)} = ... = w_n^{(k)}$ for each $k \geq 0$.

**Part (a).** By Theorems 1 and 2, there exist a unique equilibrium $(p^*, w^*)$ in the base model and a unique equilibrium $p^{c*}$ in the model with a fixed-commission rate. If $(p^*, w^*)$ is not symmetric, since all the model parameters are symmetric with respect to different platforms, customer segments, and worker types, we can find a permutation of $(p^*, w^*)$, which is not identical to $(p^*, w^*)$, but still an equilibrium, thus contradicting the uniqueness of $(p^*, w^*)$. Therefore, $(p^*, w^*)$ must be symmetric, i.e. $p_1^* = p_2^* = ... = p_n^*$ and $w_1^* = w_2^* = ... = w_n^*$. Similarly, we have $p^{c*}$ is symmetric for the model with a fixed-commission rate, i.e., $p_1^{c*} = p_2^{c*} = ... = p_n^{c*}$. This proves **part (a)**.

**Part (b).** If $\beta = w_i^*/p_i^*$, it is straightforward to check that $p_i^* = p_i^{c*}$ and $w_i^* = \beta p_i^* = \beta p_i^{c*} = w_i^{c*}$ for all $i$. Therefore, $d_i^* = d_i^{c*}$ for all $i$ as well. $\pi_i^* = (p_i^* - w_i^*)d_i^* = (p_i^{c*} - w_i^{c*})d_i^{c*} = \pi_i^{c*}$. If $\beta \neq w_i^*/p_i^*$, by the definition of the best-response operator $T(\cdot, \cdot)$, we have that, for each $k \geq 0$,

$$(p_i^{(k+1)} - w_i^{(k+1)})d_i^{(k+1)} > (p_i^{(k)} - w_i^{(k)})d_i^{(k)}.$$

Therefore,

$$\pi_i^* = (p_i^* - w_i^*)d_i^* = \lim_{k \uparrow +\infty}(p_i^{(k)} - w_i^{(k)})d_i^{(k)} > (p_i^{(0)} - w_i^{(0)})d_i^{(0)} = (p_i^{c*} - w_i^{c*})d_i^{c*} = \pi_i^{c*}.$$

This proves **part (b)**.

**Part (c).** Because $w_i^{(0)}/p_i^{(0)} = \beta < w_i^*/p_i^*$, exactly the same argument as the proof of Proposition 1(a) demonstrates that $p_i^{(1)} < p_i^{(0)}$ and $w_i^{(1)} > w_i^{(0)}$. Furthermore, by an induction argument similar to the proof of Proposition 1(a), we have if $p_i^{(k)} < p_i^{(0)}$ and $w_i^{(k)} > w_i^{(0)}$ then $p_i^{(k+1)} < p_i^{(0)}$ and $w_i^{(k+1)} > w_i^{(0)}$ for all $k \geq 1$. Putting these inequalities together and taking $k$ to limit, we have $p_i^* = \lim_{k\uparrow+\infty} p_i^{(k)} < p_i^{(0)} = p_i^{c*}$ and $w_i^* = \lim_{k\uparrow+\infty} w_i^{(k)} > w_i^{(0)} = w_i^{c*}$ for all $i$. Finally, $d_i^* > d_i^{c*}$ follows directly from $p_i^* < p_i^{c*}$. This proves **part (c)**.

**Part (c).** Because $w_i^{(0)}/p_i^{(0)} = \beta > w_i^*/p_i^*$, exactly the same argument as the proof of Proposition 1(a) demonstrates that $p_i^{(1)} > p_i^{(0)}$ and $w_i^{(1)} < w_i^{(0)}$. Furthermore, by an induction argument similar to the proof of Proposition 1(a), we have if $p_i^{(k)} > p_i^{(0)}$ and $w_i^{(k)} < w_i^{(0)}$ then $p_i^{(k+1)} > p_i^{(0)}$ and $w_i^{(k+1)} < w_i^{(0)}$ for all $k \geq 1$. Putting these inequalities together and taking $k$ to limit, we have $p_i^* = \lim_{k\uparrow+\infty} p_i^{(k)} > p_i^{(0)} = p_i^{c*}$ and $w_i^* = \lim_{k\uparrow+\infty} w_i^{(k)} < w_i^{(0)} = w_i^{c*}$ for all $i$. Finally, $d_i^* < d_i^{c*}$ follows directly from $p_i^* > p_i^{c*}$. This proves **part (d)**. □

### Proof of Corollary 1

The first part follows from the same argument as in the proof of Theorem 1. If $s_i^{s*} < d_i^{s*}$, then $P_i$ can increase its price and strictly increase its profit. If $s_i^{s*} > d_i^{s*}$, then $P_i$ can decrease its price and strictly increase its profit. As a result, under equilibrium, we must have $s_i^{s*} = d_i^{s*}$ for $i = 1, 2, ..., n$. Similarly, the equilibrium existence and uniqueness follow from the same argument as in the proof of Theorem 1. □

### Proof of Theorem 3

We first observe that the same argument as in the proof of Step I of Theorem 1 implies that, in equilibrium, the supply and demand of each platform should match. More specifically, if $\tilde{s}_i^* > \tilde{\lambda}_i^*$ (resp. $\tilde{s}_i^* < \tilde{\lambda}_i^*$), $P_i$ can decrease (resp. increase) its wage $\tilde{w}_i$ (resp. price $\tilde{p}_i$) by a sufficiently small amount to strictly increase its profit. Here, $\tilde{s}_i^*$ is the equilibrium supply of $P_i$, $\tilde{\lambda}_1^* = \tilde{d}_1^* + \tilde{d}_x^*/\tilde{n}$ is the total equilibrium demand for $P_1$'s workers, and $\tilde{\lambda}_i^* = \tilde{d}_i^*$ is the total equilibrium demand for $P_i$'s workers ($i = 2, 3, ..., n$). Using $\tilde{s}_i^* = \tilde{\lambda}_i^*$, we can write $P_i$'s profit function as follows:

$$\tilde{\pi}_i(\tilde{p}, \tilde{w}) = (\tilde{p}_i - \tilde{w}_i)\tilde{d}_i + \gamma_i\Big(\tilde{p}_x - \frac{\tilde{w}_1}{\tilde{n}}\Big)\tilde{d}_x.$$

Given $P_{-i}$'s strategy, $(\tilde{p}_{-i}, \tilde{w}_{-i})$, we use $\tilde{p}_i(\tilde{p}_{-i}, \tilde{w}_{-i})$ and $\tilde{w}_i(\tilde{p}_{-i}, \tilde{w}_{-i})$ to denote the best-response price and wage of $P_i$ under coopetition. Given $(\tilde{p}_{-i}, \tilde{w}_{-i}, \tilde{p}_x)$, the price and wage optimization of $P_1$ can be formulated as follows:

$$\begin{aligned}
\max_{(\tilde{p}_1,\tilde{w}_1,\tilde{d}_1,\tilde{d}_x)} \quad & (\tilde{p}_1 - \tilde{w}_1)\tilde{d}_1 + \gamma_1\Big(\tilde{p}_x - \frac{\tilde{w}_1}{\tilde{n}}\Big)\tilde{d}_x \\
\text{where} \quad & \sum_{j=1}^{m} \tilde{d}_{1j} = \tilde{d}_1 \\
& \tilde{p}_1 = \frac{q_{1j}}{\kappa_j} - \frac{1}{\kappa_j}\log\Big(\frac{\tilde{d}_{1j}/\Lambda_j}{1-\tilde{d}_{1j}/\Lambda_j}\Big) - \frac{1}{\kappa_j}\log\Big(1 + \sum_{i'\neq 1}\exp[\nu_j + \min\{1, \tilde{s}_{i'}/\tilde{d}_{i'}\}(q_{i'j} - \kappa_j\tilde{p}_{i'} - \nu_j)]\Big) \text{ for all } j \\
& \sum_{j=1}^{m} \tilde{d}_{xj} = \tilde{d}_x \\
& \tilde{p}_x = \frac{q_{xj}}{\kappa_j} - \frac{1}{\kappa_j}\log\Big(\frac{\tilde{d}_{xj}/\Lambda_j}{1-\tilde{d}_{xj}/\Lambda_j}\Big) - \frac{1}{\kappa_j}\log\Big(1 + \sum_{i'\neq x}\exp[\nu_j + \min\{1, \tilde{s}_{i'}/\tilde{d}_{i'}\}(q_{i'j} - \kappa_j\tilde{p}_{i'} - \nu_j)]\Big) \text{ for all } j \\
& \sum_{k=1}^{l} \tilde{s}_{1k} = \tilde{d}_1 + \frac{\tilde{d}_x}{\tilde{n}} \\
& \tilde{w}_1 = -\frac{a_{1k}}{\eta_k} - \frac{1}{\eta_{1k}}\log\Big(\frac{\tilde{s}_{1k}/\Gamma_k}{1-\tilde{s}_{1k}/\Gamma_k}\Big) + \frac{1}{\eta_k}\log\Big(1 + \sum_{i'\neq 1}\exp[\omega_k + \min\{1, \tilde{d}_{i'}/\tilde{s}_{i'}\}(a_{i'k} + \eta_k\tilde{w}_{i'} - \omega_k)]\Big) \text{ for all } k.
\end{aligned} \tag{9}$$

In addition, the price and wage optimization for $P_i$ $(i=2,3,...,n)$ can be formulated as follows (we use $\gamma_2 = 1-\gamma_1$ and $\gamma_i = 0$ for $i=3,4,...,n$):

$$\max_{(\tilde{p}_i,\tilde{w}_i,\tilde{d}_i)} \ (\tilde{p}_i - \tilde{w}_i)\tilde{d}_i + \gamma_i\Big(\tilde{p}_x - \frac{\tilde{w}_1}{\tilde{n}}\Big)\tilde{d}_x$$

$$\text{where } \sum_{j=1}^{m} \tilde{d}_{ij} = \tilde{d}_i$$

$$\tilde{p}_i = \frac{q_{ij}}{\kappa_j} - \frac{1}{\kappa_j}\log\Big(\frac{\tilde{d}_{ij}/\Lambda_j}{1-\tilde{d}_{ij}/\Lambda_j}\Big) - \frac{1}{\kappa_j}\log\Big(1+\sum_{i'\neq i}\exp[\nu_j + \min\{1,\tilde{s}_{i'}/\tilde{d}_{i'}\}(q_{i'j}-\kappa_j\tilde{p}_{i'}-\nu_j)]\Big) \text{ for all } j$$

$$\sum_{j=1}^{m} \tilde{d}_{xj} = \tilde{d}_x$$

$$\tilde{p}_x = \frac{q_{xj}}{\kappa_j} - \frac{1}{\kappa_j}\log\Big(\frac{\tilde{d}_{xj}/\Lambda_j}{1-\tilde{d}_{xj}/\Lambda_j}\Big) - \frac{1}{\kappa_j}\log\Big(1+\sum_{i'\neq x}\exp[\nu_j + \min\{1,\tilde{s}_{i'}/\tilde{d}_{i'}\}(q_{i'j}-\kappa_j\tilde{p}_{i'}-\nu_j)]\Big) \text{ for all } j$$

$$\sum_{k=1}^{l} \tilde{s}_{ik} = \tilde{d}_i$$

$$\tilde{w}_i = -\frac{a_{ik}}{\eta_k} - \frac{1}{\eta_{ik}}\log\Big(\frac{\tilde{s}_{ik}/\Gamma_k}{1-\tilde{s}_{ik}/\Gamma_k}\Big) + \frac{1}{\eta_k}\log\Big(1+\sum_{i'\neq i}\exp[\omega_k + \min\{1,\tilde{d}_{i'}/\tilde{s}_{i'}\}(a_{i'k}+\eta_k\tilde{w}_{i'}-\omega_k)]\Big) \text{ for all } k. \tag{10}$$

Following the same argument as in the proof of Step II of Theorem 1, we have that both $\tilde{p}_i(\tilde{p}_{-i},\tilde{w}_{-i})$ and $\tilde{w}_i(\tilde{p}_{-i},\tilde{w}_{-i})$ are continuously increasing in $\tilde{p}_{-i}$ and $\tilde{w}_{-i}$ for $i=1,2,...,n$. Therefore, by Tarski's Fixed Point Theorem, an equilibrium exists for the model with coopetition.

To show that the equilibrium is unique, we follow the same argument as in the proof of Lemma 5. It suffices to show that for some $k$, the $k$-fold best-response mapping, $\tilde{T}^{(k)}(\tilde{p},\tilde{w})$, (defined in a similar fashion as $T^{(k)}(\cdot,\cdot)$, but for the model with coopetition) is a contraction mapping. The same argument as in the proof of Lemma 5 implies that for any $(\tilde{p},\tilde{w})$ and $(\tilde{p}',\tilde{w}')$, we have

$$||\tilde{T}^{(k)}(\tilde{p},\tilde{w}) - \tilde{T}^{(k)}(\tilde{p}',\tilde{w}')||_1 \leq 2E^k||(p,w)-(\tilde{p}',\tilde{w}')||_1,$$

where $E<1$ is defined in the proof of Lemma 5. Consequently, $\tilde{T}^{(k^*)}$ is a contraction mapping under the $\ell_1$ norm, where $k^*$ is the smallest integer satisfying $2C^{(k^*)}<1$ (i.e., $k^* > -\log(2)/\log(E)$). The contraction mapping property of $T^{(k^*)}(\cdot,\cdot)$, as shown in the proof of Theorem 1, implies that the equilibrium is unique in the presence of coopetition, and that it can be computed using a *tatônnement* scheme. This concludes the proof of Theorem 3. □

**Proof of Theorem 4**

We first show that if $\tilde{p}_n \uparrow +\infty$, then $(\tilde{p}^*_i,\tilde{w}^*_i)$ converges to $(p^*_i,w^*_i)$ for $i=1,2...,n$. For given $(\tilde{p},\tilde{w}) = (\tilde{p}_1,\tilde{w}_1,\tilde{p}_2,\tilde{w}_2,...,\tilde{p}_3,\tilde{w}_3)$, we define the two-dimensional sequence $\{\big(\tilde{p}_i(k,j),\tilde{w}_i(k,j)\big) : 1\leq i\leq n, k\geq 1, j\geq 1\}$, where $\big(\tilde{p}(k,j),\tilde{w}(k,j)\big) = \tilde{T}^{(k)}(\tilde{p},\tilde{w})$ with $\tilde{p}_x = j$. From the proof of Lemma 5, we know that $\lim_{j\uparrow+\infty}\big(\tilde{p}(k,j),\tilde{w}(k,j)\big) = T^{(k)}(\tilde{p},\tilde{w})$.

Therefore, as shown in the proof of Theorem 3, the equilibrium strategies with $\tilde{p}_x = j$ satisfy $\big(\tilde{p}^*(j),\tilde{w}^*(j)\big) = \lim_{k\uparrow+\infty}\big(\tilde{p}(k,j),\tilde{w}(k,j)\big)$. Using the proof of Theorem 3, we have $||T^{(k)}(\tilde{p},\tilde{w}) - T^{(k)}(\tilde{p}',\tilde{w}')||_1 \leq 2E^k||(\tilde{p},\tilde{w})-(\tilde{p}',\tilde{w}')||_1$ for $k\geq 1$. Thus,

$$|\tilde{p}_i(k+1,j) - \tilde{p}_i(k,j)| \leq 2E^k||(\tilde{p}(1,j),\tilde{w}(1,j)) - (\tilde{p},\tilde{w})||_1,$$

$$|\tilde{w}_i(k+1,j) - \tilde{w}_i(k,j)| \le 2E^k ||(\tilde{p}(1,j), \tilde{w}(1,j)) - (\tilde{p}, \tilde{w})||_1.$$

As a result, $\sum_{k=1}^{+\infty} |\tilde{p}_i(k+1,j) - \tilde{p}_i(k,j)| < +\infty$ and $\sum_{k=1}^{+\infty} |\tilde{w}_i(k+1,j) - \tilde{w}_i(k,j)| < +\infty$ for $i = 1,2,...,n$. Using the dominated convergence theorem, we obtain, for all $i$,

$$\lim_{j\uparrow+\infty} \lim_{k\uparrow+\infty} (\tilde{p}_i(k,j), \tilde{w}_i(k,j)) = \lim_{k\uparrow+\infty} \lim_{j\uparrow+\infty} (\tilde{p}_i(k,j), \tilde{w}_i(k,j))$$

that is,

$$\lim_{j\uparrow+\infty} (\tilde{p}^*(j), \tilde{w}^*(j)) = \lim_{j\uparrow+\infty} \lim_{k\uparrow+\infty} (\tilde{p}(k,j), \tilde{w}(k,j)) = \lim_{k\uparrow+\infty} \lim_{j\uparrow+\infty} (\tilde{p}(k,j), \tilde{w}(k,j)) = \lim_{k\uparrow+\infty} T^{(k)}(\tilde{p}, \tilde{w}) = (p^*, w^*),$$

which states that if $\tilde{p}_x \uparrow +\infty$, then $(\tilde{p}_i^*, \tilde{w}_i^*)$ converges to $(p_i^*, w_i^*)$ for $i = 1,2,...,n$.

We next show that $\tilde{\pi}(\tilde{p}_x) := \tilde{\pi}_1(\tilde{p}^*(\tilde{p}_x), \tilde{w}^*(\tilde{p}_x)) + \tilde{\pi}_2(\tilde{p}^*(\tilde{p}_x), \tilde{w}^*(\tilde{p}_x))$ is decreasing in $\tilde{p}_x$ for sufficiently large $\tilde{p}_x$, where $(\tilde{p}_i^*(\tilde{p}_x), \tilde{w}_i^*(\tilde{p}_x))$ is the equilibrium outcome of $P_i$ under coopetition when the price of the new service is $\tilde{p}_x$.

We first show that, under a given equilibrium price and wage vector $(\tilde{p}^*, \tilde{w}^*)$ associated with $\tilde{p}_x$, the total profit of $P_1$ and $P_2$, $\tilde{\pi}(\tilde{p}_x|\tilde{p}^*, \tilde{w}^*)$ is decreasing in $\tilde{p}_n$ for sufficiently large $\tilde{p}_x$, where

$$\tilde{\pi}(\tilde{p}_x|\tilde{p}^*, \tilde{w}^*) = (\tilde{p}_1^* - \tilde{w}_1^*)\tilde{d}_1 + (\tilde{p}_2^* - \tilde{w}_2^*)\tilde{d}_2 + \Big(\tilde{p}_x - \frac{\tilde{w}_1^*}{\tilde{n}}\Big)\tilde{d}_x.$$

By Lemma 2, we have

$$\begin{aligned}\partial_{\tilde{p}_x}\tilde{\pi}(\tilde{p}_x|\tilde{p}^*, \tilde{w}^*) =& (\tilde{p}_1^* - \tilde{w}_1^*)\partial_{\tilde{p}_x}\tilde{d}_1 + (\tilde{p}_2^* - \tilde{w}_2^*)\partial_{\tilde{p}_x}\tilde{d}_2 + \tilde{d}_x + \Big(\tilde{p}_x - \frac{\tilde{w}_1^*}{\tilde{n}}\Big)\partial_{\tilde{p}_x}\tilde{d}_x \\ =& (\tilde{p}_1^* - \tilde{w}_1^*)\sum_{j=1}^m \kappa_j \bar{d}_{1j}\bar{d}_{xj}/\Lambda_j + (\tilde{p}_2^* - \tilde{w}_2^*)\sum_{j=1}^m \kappa_j \bar{d}_{2j}\bar{d}_{xj}/\Lambda_j + \tilde{d}_x \\ & - \Big(\tilde{p}_x - \frac{\tilde{w}_1^*}{\tilde{n}}\Big)\sum_{j=1}^m \kappa_j (1 - \bar{d}_{xj}/\Lambda_j)\bar{d}_{xj}.\end{aligned}$$

Hence, $\partial_{\tilde{p}_x}\tilde{\pi}(\tilde{p}_x^*|\tilde{p}^*, \tilde{w}^*) = 0$ implies that

$$\tilde{p}_x^* = (\tilde{p}_1^* - \tilde{w}_1^*)\frac{\sum_{j=1}^m \kappa_j \bar{d}_{1j}^*\bar{d}_{xj}^*/\Lambda_j}{\sum_{j=1}^m \kappa_j (1 - \bar{d}_{xj}^*/\Lambda_j)\bar{d}_{xj}^*} + (\tilde{p}_2^* - \tilde{w}_2^*)\frac{\sum_{j=1}^m \kappa_j \bar{d}_{2j}^*\bar{d}_{xj}^*/\Lambda_j}{\sum_{j=1}^m \kappa_j (1 - \bar{d}_{xj}^*/\Lambda_j)\bar{d}_{xj}^*} + \frac{\tilde{w}_1^*}{\tilde{n}}, \tag{11}$$

where $\bar{d}_{ij}^*$ is the equilibrium demand of $P_i$'s service, when $\tilde{p}_x = \tilde{p}_x^*$ satisfies Equation (11). We observe that the right-hand side of Equation (11) is decreasing with respect to $\tilde{p}_x$. Therefore, there exists a unique $\tilde{p}_x^*$ such that Equation (11) holds. Furthermore, one can check that $\partial_{\tilde{p}_x}\tilde{\pi}(\tilde{p}_x|\tilde{p}^*, \tilde{w}^*) > 0$ (resp. $< 0$) if $\tilde{p}_x < \tilde{p}_x^*$ (resp. $\tilde{p}_x > \tilde{p}_x^*$). As a result, $\tilde{\pi}(\cdot|\tilde{p}^*, \tilde{w}^*)$ is decreasing in $\tilde{p}_x$ for $\tilde{p}_x \ge \tilde{p}_x^*$. Note that $\tilde{p}_x^*$ is uniformly bounded from above by an upper bound on the right-hand side of Equation (11), say $\bar{p}^* := (p_1^* - w_1^* + p_2^* - w_2^*) + w_1^* + \frac{1}{1 - \bar{d}_0'/(\sum_j \Lambda_j)}$, where $\bar{d}_0'$ is the market share of the new joint service with $\tilde{p}_x = 0$. It then follows that, when $\tilde{p}_x \ge \bar{p}^*$, $\tilde{\pi}(\tilde{p}_x) = \tilde{\pi}(\tilde{p}_x|\tilde{p}^*(\tilde{p}_n), \tilde{w}^*(\tilde{p}_x))$ is strictly decreasing in $\tilde{p}_x$.

We observe that as $\tilde{p}_x \uparrow +\infty$, $\tilde{d}_x \downarrow 0$. Since $(\tilde{p}^*(\tilde{p}_x), \tilde{w}^*(\tilde{p}_x))$ approaches $(p^*, w^*)$ when $\tilde{p}_x \uparrow +\infty$, then $\tilde{\pi}(\tilde{p}_x) = \tilde{\pi}(\tilde{p}_x|\tilde{p}^*(\tilde{p}_x), \tilde{w}^*(\tilde{p}_x))$ approaches the equilibrium total profit of $P_1$ and $P_2$ without coopetition, that is, $\pi^* := \pi_1(p^*, w^*) + \pi_2(p^*, w^*)$. Since we have shown that $\tilde{\pi}(\cdot)$ is strictly decreasing in $\tilde{p}_x \ge \bar{p}^*$ and $\lim_{\tilde{p}_x \to +\infty} \tilde{\pi}(\tilde{p}_x) = \pi^*$, then $\tilde{\pi}^* := \max_{\tilde{p}_x} \tilde{\pi}(\tilde{p}_x) > \pi^*$, that is, the maximum total profit of $P_1$ and $P_2$ with coopetition dominates the maximum total profit without coopetition for any $\gamma \in (0,1)$. In other words, for $\tilde{p}_x \geq \bar{p}^*$, $\tilde{\pi}_1(\tilde{p}_x|\tilde{p}^*(\tilde{p}_x), \tilde{w}^*(\tilde{p}_x)) + \tilde{\pi}_2(\tilde{p}_x|\tilde{p}^*(\tilde{p}_x), \tilde{w}^*(\tilde{p}_x)) > \pi_1(p^*, w^*) + \pi_2(p^*, w^*)$. Thus, there exist a range of profit sharing parameters $(\underline{\gamma}, \bar{\gamma}) \subset (0,1)$, such that when $\gamma \in (\underline{\gamma}, \bar{\gamma})$, $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) > \pi_i(p^*, w^*)$ for $i = 1, 2$.

We next show that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) < \pi_i(p^*, w^*)$ for $i = 3, 4, ..., n$. Since $\lim_{\tilde{p}_x \uparrow +\infty} \tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) = \pi_i(p^*, w^*)$ for $i = 3, 4, ..., n$, it suffices to show that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*)$ is increasing in $\tilde{p}_x$. To show this monotonicity result, we prove that, for any $k$, $(\tilde{p}_i(k,j) - \tilde{w}_i(k,j))\tilde{d}_i(k,j)$ is increasing in $j$ for $i = 3, 4, ..., n$, where $\tilde{p}_i(k,j)$ and $\tilde{w}_i(k,j)$ are defined above, and $\tilde{d}_i(k,j)$ is the associated demand (and supply) for $P_i$ in round $k$ of the *tatônnement* scheme. By the proof of Lemma 4, for each $k$, both the profit margin $\tilde{m}_i(k,j) := \tilde{p}_i(k,j) - \tilde{w}_i(k,j)$ and the demand $\tilde{d}_i(k,j)$ are increasing in the price of the new service $\tilde{p}_x = j$, and so is $(\tilde{p}_i(k,j) - \tilde{w}_i(k,j))\tilde{d}_i(k,j)$. Taking $k$ to infinity, we obtain that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) = \lim_{k \to \infty}(\tilde{p}_i(k,j) - \tilde{w}_i(k,j))\tilde{d}_i(k,j)$ is increasing in $\tilde{p}_x = j$. This concludes the proof of Theorem 4. □

**Proof of Proposition 3**

By Theorem 4, we can select $\gamma_0 \in (\underline{\gamma}, \bar{\gamma})$ and $\tilde{p}_x^* = \arg\max_{\tilde{p}_x}\{\pi_1(\tilde{p}^*, \tilde{w}^*) + \pi_2(\tilde{p}^*, \tilde{w}^*)\}$ that maximize the total profit of both platforms, so that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*|\tilde{p}_x^*, \gamma_0) > \pi_i(p^*, w^*)$, for $i = 1, 2$. Thus, for any $\theta_1 + \theta_2 = 1$ $(\theta_i > 0)$, $(\tilde{p}_x^*, \gamma_0)$ is a feasible solution to the optimization problem in (3). Therefore, an optimal solution to (3), $(\tilde{p}_x^{**}, \gamma^{**})$, exists and satisfies the following:

$$
\begin{aligned}
&(\tilde{\pi}_1(\tilde{p}^*, \tilde{w}^*|\tilde{p}_x^{**}, \gamma^{**}) - \pi_1(p^*, w^*))^{\theta_1} \cdot (\tilde{\pi}_2(\tilde{p}^*, \tilde{w}^*|\tilde{p}_x^{**}, \gamma^{**}) - \pi_2(p^*, w^*))^{\theta_2} \\
\geq &(\tilde{\pi}_1(\tilde{p}^*, \tilde{w}^*|\tilde{p}_n^*, \gamma_0) - \pi_1(p^*, w^*))^{\theta_1} \cdot (\tilde{\pi}_2(\tilde{p}^*, \tilde{w}^*|\tilde{p}_x^*, \gamma_0) - \pi_2(p^*, w^*))^{\theta_2} > 0.
\end{aligned}
$$

As a result, we have $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*|\tilde{p}_x^{**}, \gamma^{**}) > \pi_i(p^*, w^*)$ for $i = 1, 2$. Finally, by the proof of Theorem 4, we have that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*)$ $(i = 3, 4, ..., n)$ is increasing in $\tilde{p}_x$, which, together with $\lim_{\tilde{p}_x \uparrow +\infty} \tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) = \pi_i(p^*, w^*)$, implies that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) < \pi_i(p^*, w^*)$ for all $i = 3, 4, ..., n$. This concludes the proof of Proposition 3. □

**Proof of Proposition 4**

**Part (a)**. Denote $(\tilde{p}_i(k,j), \tilde{w}_i(k,j) : 1 \leq i \leq n)$ as the price and wage of each platform's original service under the price of the new service $\tilde{p}_x = j$. By Equations (9) and (10), as $r \uparrow +\infty$, $\tilde{w}_i(k,j) \uparrow +\infty$ for all $i = 1, 2, ..., n$, $k = 1, 2, ...$, and $j > 0$. Then, by taking $k \uparrow +\infty$, we have that, for any $\tilde{p}_x$, the equilibrium wage of $P_1$, $\tilde{w}_1^* \uparrow +\infty$. By (11), we must have $\lim_{r \uparrow +\infty} \tilde{p}_x^* = +\infty$. To show that $\lim_{r \uparrow +\infty} \tilde{p}_x^{**} \uparrow +\infty$, we note that $\lim_{r \uparrow +\infty} \tilde{w}_1^*/\tilde{n} = +\infty$. Under the Nash Bargaining equilibrium, we must have $\tilde{p}_x > \tilde{w}_1/\tilde{n}$, which together with $\lim_{r \uparrow +\infty} \tilde{w}_1^*/\tilde{n} = +\infty$ leads to $\lim_{r \uparrow +\infty} \tilde{p}_x^{**} \uparrow +\infty$. This concludes the proof of Part (a).

**Part (b)**. We next show that the total profit under coopetition increases when $\tilde{p}_x = \bar{p}$ and $r$ is sufficiently small. Note that, as $r \downarrow 0$, by Equations (9) and (10), $\tilde{w}_i(k,j) \downarrow 0$ for all $i = 1, 2, ..., n$, $k = 1, 2, ...$, and $j > 0$. Then, by taking $k \uparrow +\infty$, we have that, for any $\tilde{p}_x$, the equilibrium wage of $P_1$, $\tilde{w}_1^* \downarrow 0$. Therefore, for $\tilde{p}_x = \bar{p}$, the equilibrium profit from the new service $(\tilde{p}_x - \tilde{w}_1^*/\tilde{n})\tilde{d}_x^* > 0$. This implies that the total profit under coopetition increases when $\tilde{p}_x = \bar{p}$ and $r$ is sufficiently small. Consequently, we can find a profit sharing parameter $\gamma$ such that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*|\bar{p}, \gamma) > \pi_i(p^*, w^*)$ for $i = 1, 2$. This concludes the proof of Part (b-i).

Finally, we show Part (b-ii). Specifically, we prove that if there is a finite upper bound on the price of the new service set by the platforms, i.e., $\tilde{p}_x \leq \bar{p}$, at least one platform would be worse off under coopetition, namely, either $\tilde{\pi}_1(\tilde{p}^*, \tilde{w}^*) < \pi_1(p^*, w^*)$ or $\tilde{\pi}_2(\tilde{p}^*, \tilde{w}^*) < \pi_2(p^*, w^*)$, when $r$ is sufficiently large. By the proof of

Part (a), as $r \uparrow +\infty$, we have $\tilde{w}_1^* \uparrow +\infty$ for any $\tilde{p}_x$. Since $\tilde{p}_x \le \bar{p}$, the profit from the new joint service is such that $(\tilde{p}_x - \tilde{w}_1^*/\tilde{n})\tilde{d}_x < 0$.

Furthermore, in the presence of coopetition, $P_i$ needs to charge a lower price relative to the setting without coopetition in order to induce the same demand assuming that its competitor offers the same price. Thus, for any $(p_{-i}, w_{-i})$, $P_i$'s optimal profit from its original service is lower under coopetition. By taking the index of the best-response mapping $k$ to infinity, we have that $P_i$'s equilibrium profit from its original service is lower under coopetition for $i = 1, 2$. Since we have shown that for a sufficiently large $r$, the total profit from the new service $(\tilde{p}_x - \tilde{w}_1^*/\tilde{n})\tilde{d}_x$ is negative, then the total profit of $P_1$ and $P_2$ is lower under coopetition:

$$\tilde{\pi}_1^* + \tilde{\pi}_2^* = (\tilde{p}_1^* - \tilde{w}_1^*)\tilde{d}_1^* + (\tilde{p}_2^* - \tilde{w}_2^*)\tilde{d}_2^* + (\tilde{p}_x - \tilde{w}_1^*/\tilde{n})\tilde{d}_x < (p_1^* - w_1^*)d_1^* + (p_2^* - w_2^*)d_2^* = \pi_1^* + \pi_2^*.$$

Consequently, when $r$ is sufficiently large, at least one of the platforms is worse off for any $\gamma$, and this concludes the proof of Proposition 4. □

**Proof of Proposition 5**

We first show that the total profit under coopetition increases when $\tilde{p}_x = \bar{p}$ and $q_3$ is sufficiently large. Recall that given $(\tilde{p}_i, \tilde{w}_i)$ for $i = 1, 2$ and $(\tilde{p}_x, \gamma)$, the price and wage optimization of $P_3$ can be formulated as follows:

$$\max_{(\tilde{p}_3, \tilde{w}_3, \tilde{d}_3)} \; (\tilde{p}_3 - \tilde{w}_3)\tilde{d}_3$$

$$\begin{aligned}
\text{where } & \sum_{j=1}^{m} \tilde{d}_{3j} = \tilde{d}_3 \\
& \tilde{p}_3 = \frac{q_3 \iota_j}{\kappa_j} - \frac{1}{\kappa_j}\log\left(\frac{\tilde{d}_{3j}/\Lambda_j}{1 - \tilde{d}_{3j}/\Lambda_j}\right) - \frac{1}{\kappa_j}\log\left(1 + \sum_{i' \neq 3} \exp[\nu_j + \min\{1, \tilde{s}_{i'}/\tilde{d}_{i'}\}(q_{i'j} - \kappa_j \tilde{p}_{i'} - \nu_j)]\right) \; \forall j \\
& \sum_{j=1}^{m} \tilde{d}_{xj} = \tilde{d}_x \\
& \tilde{p}_x = \frac{q_{xj}}{\kappa_j} - \frac{1}{\kappa_j}\log\left(\frac{\tilde{d}_{xj}/\Lambda_j}{1 - \tilde{d}_{xj}/\Lambda_j}\right) - \frac{1}{\kappa_j}\log\left(1 + \sum_{i' \neq x} \exp[\nu_j + \min\{1, \tilde{s}_{i'}/\tilde{d}_{i'}\}(q_{i'j} - \kappa_j \tilde{p}_{i'} - \nu_j)]\right) \; \forall j \\
& \sum_{k=1}^{l} \tilde{s}_{3k} = \tilde{d}_3 \\
& \tilde{w}_3 = -\frac{a_3 \psi_k}{\eta_k} + \frac{1}{\eta_{3k}}\log\left(\frac{\tilde{s}_{3k}/\Gamma_k}{1 - \tilde{s}_{3k}/\Gamma_k}\right) + \frac{1}{\eta_k}\log\left(1 + \sum_{i' \neq 3} \exp[\omega_k + \min\{1, \tilde{d}_{i'}/\tilde{s}_{i'}\}(a_{i'k} + \eta_k \tilde{w}_{i'} - \omega_k)]\right) \; \forall k.
\end{aligned} \tag{12}$$

It follows from the optimization problem in (12) that, given $(\tilde{p}_i, \tilde{w}_i)$ for $i = 1, 2$ and $(\tilde{p}_x, \tilde{n})$, if we take $q_3 \uparrow +\infty$, the best responses of $P_3$ will satisfy $\tilde{p}_3 \uparrow +\infty$ and $\tilde{d}_{3j} \uparrow \Lambda_j$ for all $j$. Consequently, as $q_3 \uparrow +\infty$, $\tilde{d}_{1j} \downarrow 0$ and $\tilde{d}_{2j} \downarrow 0$ for all $j$. Since supply equals demand, we have $\tilde{s}_{1k} \downarrow 0$ and $\tilde{s}_{2k} \downarrow 0$ for all $k$, which imply that $w_i^* \downarrow 0$ for $i = 1, 2$. Therefore, for $\tilde{p}_x = \bar{p}$, the equilibrium profit from the new service is such that $(\tilde{p}_x - \tilde{w}_1^*/\tilde{n})\tilde{d}_x^* > 0$. Similarly, for the model without coopetition, as $q_3 \uparrow +\infty$, $d_{1j}^* \downarrow 0$ and $d_{2j}^* \downarrow 0$ for all $j$ under equilibrium. So $d_1^* = \sum_j d_{1j}^*$ and $d_2^* = \sum_j d_{2j}^*$ will both decrease to 0 as $q_3 \downarrow 0$. Therefore, the profit of $P_i$ without coopetition, $(p_i^* - w_i^*)d_i^*$ will decrease to 0 as $q_3 \uparrow +\infty$. This implies that the total profit of $P_1$ and $P_2$ under coopetition will increase when $\tilde{p}_x = \bar{p}$ and $q_3$ is sufficiently large. Consequently, we can find a profit sharing parameter $\gamma$ and a price for the joint new service $\tilde{p}_x \le \bar{p}$, such that $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*|\bar{p}, \gamma) > \pi_i(p^*, w^*)$ for $i = 1, 2$. This concludes the proof of the first part.

We next show that the total profit under coopetition will decrease for all $\tilde{p}_x \le \bar{p}$ and when $a_3$ is sufficiently large. It follows from (12) that, given $(\tilde{p}_i, \tilde{w}_i)$ for $i = 1, 2$ and $(\tilde{p}_x, \tilde{n})$, if we take $a_3 \uparrow +\infty$, $\tilde{w}_3 \downarrow 0$ and $\tilde{s}_{3k} \uparrow \Gamma_k$ for all $k$. Then, for $P_i$ $(i = 1, 2)$, the wage $\tilde{w}_i$ satisfies $\tilde{w}_i = -\frac{a_{ik}}{\eta_k} + \frac{1}{\eta_{ik}} \log\left(\frac{\tilde{s}_{ik}/\Gamma_k}{1-\tilde{s}_{ik}/\Gamma_k}\right) + \frac{1}{\eta_k} \log\Big(1 + \sum_{i' \neq i} \exp[\omega_k + \min\{1, \tilde{d}_{i'}/\tilde{s}_{i'}\}(a_{i'k} + \eta_k \tilde{w}_{i'} - \omega_k)]\Big)$ for all $k$. Since $a_{3k} = a_3 \psi_k$ increases to $+\infty$ as $a_3 \uparrow \infty$, then $\tilde{w}_i$ will increase to $+\infty$ as $a_3 \uparrow +\infty$. Thus, since $\tilde{p}_x \le \bar{p} < +\infty$, the profit margin of the new joint service is negative when $a_3$ is sufficiently large, that is, $\tilde{p}_x - \tilde{w}_1^*/\tilde{n} < 0$.

In the presence of coopetition, $P_i$ needs to charge a lower price relative to the setting without coopetition in order to induce the same demand assuming that its competitor offers the same price. As a result, for any $(p_{-i}, w_{-i})$, $P_i$'s optimal profit from its original service is lower under coopetition. In particular, under equilibrium, $P_i$'s profit from its original service is lower in the presence of coopetition relative to the setting without coopetition for $i = 1, 2$. Since we have shown that for a sufficiently large $a_3$, the total profit from the new service is negative, then the total profit of $P_1$ and $P_2$ is lower under coopetition, that is,

$$\tilde{\pi}_1^* + \tilde{\pi}_2^* = (\tilde{p}_1^* - \tilde{w}_1^*)\tilde{d}_1^* + (\tilde{p}_2^* - \tilde{w}_2^*)\tilde{d}_2^* + (\tilde{p}_x - \tilde{w}_1^*/\tilde{n})\tilde{d}_x < (p_1^* - w_1^*)d_1^* + (p_2^* - w_2^*)d_2^* = \pi_1^* + \pi_2^*.$$

Consequently, at least one of the platforms is worse off for any $\gamma$, when $a_3$ is sufficiently large, and this concludes the proof of Proposition 5. □

**Proof of Proposition 6**

First, since $d_i^* = s_i^*$ without coopetition and $\tilde{\lambda}_i^* = \tilde{s}_i^*$ with coopetition for $i = 1, 2, ..., n$, we have

$$RS^* = \sum_{j=1}^{m} \frac{\Lambda_j}{\kappa_j} \log\left(1 + \sum_{i=1}^{n} \exp(q_{ij} - \kappa_j p_i^*)\right)$$

and

$$\tilde{RS}^* = \sum_{j=1}^{m} \frac{\Lambda_j}{\kappa_j} \log\left(1 + \exp(q_{xj} - \kappa_j \tilde{p}_x) + \sum_{i=1}^{n} \exp(q_{ij} - \kappa_j \tilde{p}_i^*)\right).$$

We observe that if $\tilde{p}_i^* \le p_i^*$ for $i = 1, 2, ..., n$, then we have

$$\begin{aligned}
\tilde{RS}^* &= \sum_{j=1}^{m} \frac{\Lambda_j}{\kappa_j} \log\left(1 + \exp(q_{xj} - \kappa_j \tilde{p}_x) + \sum_{i=1}^{n} \exp(q_{ij} - \kappa_j \tilde{p}_i^*)\right) \\
&> \sum_{j=1}^{m} \frac{\Lambda_j}{\kappa_j} \log\left(1 + \sum_{i=1}^{n} \exp(q_{ij} - \kappa_j \tilde{p}_i^*)\right) \\
&\ge \sum_{j=1}^{m} \frac{\Lambda_j}{\kappa_j} \log\left(1 + \sum_{i=1}^{n} \exp(q_{ij} - \kappa_j p_i^*)\right) = RS^*.
\end{aligned}$$

Consequently, it suffices to show that $\tilde{p}_i^* \le p_i^*$ for $i = 1, 2, ..., n$.

We define $(\tilde{p}^*(k, \tilde{p}_x), \tilde{w}^*(k, \tilde{p}_x)) := \tilde{T}^{(k)}(p^*, w^*)$, where $\tilde{T}^{(k)}(\cdot, \cdot)$ is the $k$-fold best-response mapping when the price of the new service is $\tilde{p}_x$. Then, the corresponding price and wage for $P_i$ are given by $(\tilde{p}_i^*(k, \tilde{p}_x), \tilde{w}_i^*(k, \tilde{p}_x))$. On the other hand, we know that $(p^*, w^*) = T^{(k)}(p^*, w^*)$ for any $k \ge 1$, where $T^{(k)}(\cdot, \cdot)$ is the $k$-fold best-response mapping of the model without coopetition, which can also be viewed as a special case of $\tilde{T}^{(k)}(\cdot, \cdot)$ with $\tilde{p}_x = +\infty$. Comparing the best-response formulations of $\tilde{T}^{(1)}$ and $T^{(1)}$ (see the proof of Theorems 1 and 3), one can show that given $(p_{-i}^*, w_{-i}^*)$, the best-response price $\tilde{p}_i^*(1, \tilde{p}_x)$ is increasing in $\tilde{p}_x$. Since the model without coopetition can be viewed as a special case of the model with coopetition when $\tilde{p}_x = +\infty$, we

have $\tilde{p}_i^*(1,\tilde{p}_x) < \tilde{p}_i^*(1,+\infty) = p_i^*$ for all $i = 1,2,...,n$. Then, by following the same argument as in the proof of Theorem 3, we conclude that $\tilde{p}_i^*(k+1,\tilde{p}_x)$ is strictly increasing in both $\tilde{p}_x$ and $\tilde{p}_{i'}^*(k)$ for $i = 1,2,...,n$, $i' \neq i$, and for any $k$. Using a standard induction argument, we obtain $\tilde{p}_i^*(k,\tilde{p}_x) < \tilde{p}_i^*(k,+\infty) = p_i^*$ for $k \geq 1$ and $i = 1,2,...,n$. Thus, $\tilde{p}_i^* = \lim_{k\uparrow+\infty} \tilde{p}_i^*(k,\tilde{p}_x) < p_i^*$ for $i = 1,2,...,n$, and this concludes the proof of Proposition 6.
□

**Proof of Proposition 7**

First, we highlight that for the model without coopetition $s_i^* = d_i^*$ for $i = 1,2,...,n$, whereas for the model with coopetition $\tilde{s}_i^* = \tilde{\lambda}_i^*$ for $i = 1,2,...,n$. We have

$$DS_i^* = \sum_{k=1}^{l} \frac{\Gamma_k}{\eta_k} \log\left[1 + \exp(a_{ik} + \eta_k w_i^*)\right],\ i = 1,2,...,n,$$

and

$$\tilde{DS}_i^* = \sum_{k=1}^{l} \frac{\Gamma_k}{\eta_k} \log\left[1 + \exp(a_{ik} + \eta_k \tilde{w}_i^*)\right],\ i = 1,2,...,n.$$

We next show the first part. Specifically, we show the following three claims: (a) if $\tilde{n} = 1$, then $\tilde{w}_1^* > w_1^*$; (b) if $\tilde{n}$ is sufficiently large, then $\tilde{w}_1^* < w_1^*$; and (c) $\tilde{w}_1^*$ is continuously decreasing in $\tilde{n}$. Then, Claims (a), (b), and (c) would imply the first part of Proposition 7.

**Claim (a):** If $\tilde{n} = 1$, from the proof of Theorem 3, we have $\tilde{s}_1^* = \tilde{\lambda}_1^* = \tilde{d}_1^* + \tilde{d}_x^*/\tilde{n} = \tilde{d}_1^* + \tilde{d}_x^*$. As shown in the proof of Proposition 6, $\tilde{p}_1^* < p_1^*$, and hence $\tilde{s}_1^* = \tilde{d}_1^* + \tilde{d}_x^* > d_1^* = s_1^*$. This implies that $\tilde{w}_1^* > w_1^*$ and concludes the proof of Claim (a).

**Claim (b):** As $\tilde{n} \uparrow +\infty$, we have $\tilde{s}_1^* = \tilde{\lambda}_1^* = \tilde{d}_1^* + \tilde{d}_x^*/\tilde{n} = \tilde{d}_1^*$. We next show that $\tilde{d}_1^* < d_1^*$. As in the proof of Proposition 6, for any $(\tilde{p}_x, \gamma)$, we define $(\tilde{p}^*(k,\tilde{p}_x), \tilde{w}^*(k,\tilde{p}_x)) := \tilde{T}^{(k)}(p^*, w^*)$, where $\tilde{T}^{(k)}(\cdot,\cdot)$ is the $k$-fold best-response mapping when the price of the new service is $\tilde{p}_x$. Then, the corresponding price and wage for $P_i$ are given by $(\tilde{p}_i^*(k,\tilde{p}_x), \tilde{w}_i^*(k,\tilde{p}_x))$. On the other hand, we know that $(p^*, w^*) = T^{(k)}(p^*, w^*)$ for any $k \geq 1$, where $T^{(k)}(\cdot,\cdot)$ is the $k$-fold best-response mapping of the model without coopetition, which can also be viewed as a special case of $\tilde{T}^{(k)}(\cdot,\cdot)$ with $\tilde{p}_x = +\infty$. By comparing the best-response formulations of $\tilde{T}^{(1)}$ and $T^{(1)}$ (see the proof of Theorems 1 and 3), one can show that given $(p_{-1}^*, w_{-1}^*)$, the best-response demand $\tilde{d}_1^*(1,\tilde{p}_x)$ is increasing in $\tilde{p}_x$. Since the model without coopetition can be viewed as a special case of the model with coopetition with $\tilde{p}_x = +\infty$, we have $\tilde{d}_1^*(1,\tilde{p}_x) < \tilde{d}_1^*(1,+\infty) = d_1^*$. Then, by following the same argument as in the proof of Theorem 3, we conclude that $\tilde{d}_1^*(k+1,\tilde{p}_x)$ is strictly increasing in $\tilde{p}_x$ for $k \geq 1$. Using an induction argument, we obtain $\tilde{d}_1^* = \lim_{k\uparrow+\infty} \tilde{d}_1^*(k,\tilde{p}_x) < d_1^*$. Thus, $\tilde{s}_1^* = \tilde{d}_1^* < d_1^* = s_1^*$. This implies that $\tilde{w}_1^* < w_1^*$ and concludes the proof of Claim (b).

**Claim (c):** We show that $\tilde{w}_i^*$ is decreasing in $\tilde{n}$ for any $i = 1,2,...,n$. We define $(\tilde{p}^*(k,\tilde{n}), \tilde{w}^*(k,\tilde{n})) := \tilde{T}^{(k)}(p^*, w^*)$, where $\tilde{T}^{(k)}(\cdot,\cdot)$ is the $k$-fold best-response mapping when the price of the new service is $\tilde{p}_x$ and the pooling parameter is $\tilde{n}$. By examining the best-response mapping $\tilde{T}^{(1)}$ (see the proof of Theorem 4), we obtain that given $(p_{-i}^*, w_{-i}^*)$, $\tilde{w}_i^*(1,\tilde{n})$ is decreasing in $\tilde{n}$ for $i = 1,2,...,n$. Furthermore, the best-response mapping is increasing in $\tilde{w}_{-i}^*$ (see the proof of Theorem 1). Using an induction argument, we obtain that $\tilde{w}_i^*(k,\tilde{n})$ is increasing in $\tilde{w}_{-i}^*(k-1,\tilde{n})$, which is decreasing in $\tilde{n}$. Thus, $\tilde{w}_i^*(k,\tilde{n})$ is decreasing in $\tilde{n}$ for $k \geq 1$ and for $i = 1,2,...,n$. As a result, the equilibrium wage under coopetition $\tilde{w}_i^* = \lim_{k\uparrow+\infty} \tilde{w}_i^*(k,\tilde{n})$ is decreasing in $\tilde{n}$ for $i=1,2,...,n$. This concludes the proof of Claim (c). Claims (a), (b), and (c) together imply that Proposition 7(a) holds.

We next show the second part of the proposition. The same argument as the proof of Claim (b) above implies that $\tilde{w}_i^* < w_i^*$ for $i=2,3,...,n$, so we must have

$$\tilde{DS}_i^* = \sum_{k=1}^{l} \frac{\Gamma_k}{\eta_k} \log\left[1+\exp(a_{ik}+\eta_k \tilde{w}_i^*)\right] < \sum_{k=1}^{l} \frac{\Gamma_k}{\eta_k} \log\left[1+\exp(a_{ik}+\eta_k w_i^*)\right] = DS_i^* \text{ for all } i=2,3,...,n.$$

This concludes the proof of part (b). □

### Proof of Proposition 8

Following the same argument as in the proof of Theorem 4, we know that if $\tilde{p}_x \to +\infty$, then $\lim_{\tilde{p}_x \uparrow +\infty}(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = (p_1^*, w_1^*, p_2^*, w_2^*)$, $\lim_{\tilde{p}_x \uparrow +\infty}(\tilde{d}_1^*, \tilde{d}_2^*) = (d_1^*, d_2^*)$, and $\lim_{\tilde{p}_x \uparrow +\infty}(\tilde{s}_1^*, \tilde{s}_2^*) = (s_1^*, s_2^*)$. Therefore, we have $\lim_{\tilde{p}_n \uparrow +\infty} \tilde{\pi}_i^* + \tilde{DS}_i^* = \pi_i^* + DS_i^*$ for $i=1,2$.

We next show that $\tilde{R}_i(\tilde{p}_x) := \tilde{\pi}_i^* + \tilde{DS}_i^*$ $(i=1,2)$ is decreasing in $\tilde{p}_x$ for a sufficiently large $\tilde{p}_n$, where $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, w_2^*)$ is the equilibrium under coopetition with $\tilde{p}_x$. Given the equilibrium price and wage vector $(\tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$, we define the total platform and driver surplus of both platforms as follows:

$$\begin{aligned}
\tilde{R}(\tilde{p}_x | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) =& \tilde{R}_1(\tilde{p}_x) + \tilde{R}_2(\tilde{p}_x) \\
=& (\tilde{p}_1^* - \tilde{w}_1^*)\tilde{d}_1^* + (\tilde{p}_2^* - \tilde{w}_2^*)\tilde{d}_2^* + (\tilde{p}_x - \tilde{w}_1^*/\tilde{n})\tilde{d}_x \\
&+ \sum_{k=1}^{l} \frac{\Gamma_k}{\eta_k} \log\left[1+\exp(a_{1k}+\eta_k \tilde{w}_1^*)\right] + \sum_{k=1}^{l} \frac{\Gamma_k}{\eta_k} \log\left[1+\exp(a_{2k}+\eta_k \tilde{w}_2^*)\right],
\end{aligned}$$

where $\tilde{s}_1^* = \tilde{d}_1^* + \tilde{d}_x/\tilde{n}$ and $\tilde{s}_2^* = \tilde{d}_2^*$. Following the same argument as in the proof of Theorem 4, we have $\partial_{\tilde{p}_x} R(\tilde{p}_x | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) < 0$ for a sufficiently large $\tilde{p}_x$. This also shows that $R(\tilde{p}_x | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ is strictly decreasing in $\tilde{p}_x$ for a sufficiently large $\tilde{p}_x$. We have also shown that $\lim_{\tilde{p}_x \uparrow +\infty} \tilde{R}(\tilde{p}_x | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = \lim_{\tilde{p}_x \uparrow +\infty}(\tilde{\pi}_1^* + \tilde{DS}_1^* + \tilde{\pi}_2^* + \tilde{DS}_2^*) = \pi_1^* + DS_1^* + \pi_2^* + DS_2^*$. Since $\tilde{R}(\cdot | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*)$ is strictly decreasing in $\tilde{p}_x$ for a sufficiently large $\tilde{p}_x$, one can find a value of $\tilde{p}_x$ such that $\tilde{R}(\tilde{p}_x | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) > \pi_1^* + DS_1^* + \pi_2^* + DS_2^*$. Since $\tilde{R}(\tilde{p}_x | \tilde{p}_1^*, \tilde{w}_1^*, \tilde{p}_2^*, \tilde{w}_2^*) = \tilde{\pi}_1^* + \tilde{DS}_1^* + \tilde{\pi}_2^* + \tilde{DS}_2^*$, one can find a value of $\gamma$ such that, under the price of the new service $\tilde{p}_x$, $\tilde{\pi}_i^* + \tilde{DS}_i^* > \pi_i^* + DS_i^*$ for $i=1,2$. By Theorem 4 and Proposition 7, for any $(\tilde{p}_x, \gamma)$, $\tilde{\pi}_i(\tilde{p}^*, \tilde{w}^*) < \pi_i(p^*, w^*)$ and $\tilde{DS}_i^* < DS_i^*$ for all $i=3,4,...,n$. Hence, $\tilde{\pi}_i^* + \tilde{DS}_i^* < \pi_i^* + DS_i^*$ for all $i=3,4,...,n$. This concludes the proof of Proposition 8. □

**Proof of Theorem 5**

Since $\kappa(0+) = +\infty$, we must have $s_i^{e*} > d_i^{e*}$ for $i = 1, 2$ under equilibrium. Hence, $P_i$'s profit under equilibrium can be written as $[f_i - \kappa(s_i - d_i) - w_i]d_i$. Given $(p_{-i}, w_{-i})$, we rewrite $P_i$'s profit as a function of $d_i$ and $s_i$:

$$\begin{aligned}
&\max_{(f_i,w_i,d_i,s_i)} \pi_i^e(f_i, w_i, s_i, d_i | f_{-i}, w_{-i}) \\
&\quad \text{where } \pi_i^e(f_i, w_i, s_i, d_i | f_{-i}, w_{-i}) = (f_i - \kappa(s_i - d_i) - w_i)d_i \\
&\qquad \sum_{j=1}^{m} d_{ij} = d_i \\
&\qquad p_i = \frac{q_{ij}}{\kappa_j} - \frac{1}{\kappa_j}\log\Big(\frac{d_{ij}/\Lambda_j}{1 - d_{ij}/\Lambda_j}\Big) - \frac{1}{\kappa_j}\log\Big(1 + \sum_{i' \neq i}\exp(q_{i'j} - \kappa_j p_{i'})\Big) \ \forall j \qquad (13) \\
&\qquad \sum_{k=1}^{l} s_{ik} = s_i \\
&\qquad w_i = -\frac{a_{ik}}{\eta_k} + \frac{1}{\eta_{ik}}\log\Big(\frac{s_{ik}/\Gamma_k}{1 - s_{ik}/\Gamma_k}\Big) + \frac{1}{\eta_k}\log\Big(1 + \sum_{i' \neq i}\exp[\omega_k + \frac{d_{i'}}{s_{i'}}(a_{i'k} + \eta_k w_{i'} - \omega_k)]\Big) \ \forall k.
\end{aligned}$$

Hence, given $d_i$, there exists a unique price $f_i$ such that all the constraints in (13) hold, which we denote as $f_i(d_i)$. Analogously, given $s_i$, there exists a unique wage $w_i$ such that all the constraints in (13) hold, which we denote as $w_i(s_i)$. Thus, given $(f_{-i}, w_{-i})$, $P_i$'s best response can be characterized as follows:

$$\begin{aligned}
&\max_{(d_i,s_i)} \pi_i^e(f_i(d_i), w_i(s_i), d_i, s_i | f_{-i}, w_{-i}) \\
&\quad s.t.\ d_i < s_i.
\end{aligned}$$

Given $P_i$'s demand, $d_i$, the best-response supply of $P_i$ should be $\arg\max_{s > d_i}[-w_i(s) + \kappa(s - d_i)]$. As a result, we can reduce $\pi_i^e(f_i(d_i), w_i(s_i), d_i, s_i | f_{-i}, w_{-i})$ to the single-variable function $\pi_i^e(d_i | f_{-i}, w_{-i}) = (f_i(d_i) - h_i(d_i))d_i$, where $h(d_i) := \max_{s > d_i}[-w_i(s) + \kappa(s - d_i)]$.

We denote by $(f_i^e(f_{-i}, w_{-i}), w_i^e(f_{-i}, w_{-i}))$ $P_i$'s best-response price and wage functions given $(f_{-i}, w_{-i})$. Following the same argument as in Step II of the proof of Theorem 1, we can show that $(f_i^e(f_{-i}, w_{-i}), w_i^e(f_{-i}, w_{-i}))$ is continuously increasing in $f_{-i}$ and $w_{-i}$. Thus, an equilibrium $(f^{e*}, w^{e*})$ exists.

To show that the equilibrium is unique, we denote by $T_e(\cdot, \cdot)$ the best-response mapping of the model with endogenous waiting times, that is, $T_e(f, w) = \big(f_i^e(f_{-i}, w_{-i}), w_i^e(f_{-i}, w_{-i}) : 1 \le i \le n\big)$. Using the same argument as in the proof of Lemma 5, we obtain that there exists a constant $C = \max\big\{\frac{\exp(q_i)}{1+\exp(q_i)}, \frac{\exp(a_i)}{1+\exp(a_i)} : i = 1, 2, ..., n\big\} \in (0, 1)$, such that

$$||T_e^{(k)}(f, w) - T_e^{(k)}(f', w')||_1 \le 2C^{(k)}||(f, w) - (f', w')||_1,$$

and hence the $k^*$-fold best-response mapping, $T_e^{(k^*)}(\cdot, \cdot)$, is a contraction mapping, where $k^* > -\log(2)/\log(C)$. Consequently, using the same argument as in the proof of Lemma 5, the equilibrium is unique and can be computed using a *tatônnement* scheme. This concludes the proof of Theorem 5. ☐